GRAPH THEORY PART II

BEGINNER CIRCLE 4/24/2016

1. Finishing Up From Last Week

Adrian has an obsession with sorting things. He is having a party next week, and decides that he will rank the 50 guests by popularity. He will say that one person is more popular if they have more friends at the party. However, he finds that this is impossible, no matter who he invites and how many friends they have. This is because there are always at least two people with the same number of friends. To help him out, we will show that what he is trying to do is impossible by proving the following statement:

There must be at least two people with the same number of friends.

(1) For a proof by contradiction, we first assume the opposite statement is true. What is the opposite of the statement:

There are two people at the party with the same number of friends at the party?

Use full sentences for your answer.

Each person has a unique (different) number of friends at the party.

(2) Can you turn this problem into one about graphs? What are the vertices? What about the edges?

Each person is represented by a vertex and a friendship between two people is represented by the edge between two vertices

(3) Let X be the number of friends (at the party) a person at the party has. What are the possible values of X?

$X = 0$ to 49

(4) Why does this tell us that there is a person who has 49 friends? Why does this also show that someone at the party has no friends? Explain in full sentences.

Since X can take on 50 values and there are 50 people, each with a different number of friends, each of the 50 people must be assigned to a person at the party. In particular, someone is assigned 49 and someone else is assigned 0.

(5) Why is it that the person with 49 friends is friends with everybody in the room?

There are 50 people in the room and someone is friends with 49 people (the 50th person being themselves) so they're friends with everyone.

(6) Why is it impossible for someone to be friends with everybody in the room?

We also showed that someone at the party has 0 friends at the party, so it's impossible for the person with 49 friends to be friends with everyone (as they are not friends with the person with 0 friends).

(7) Conclude that there are two people that have the same number of friends.

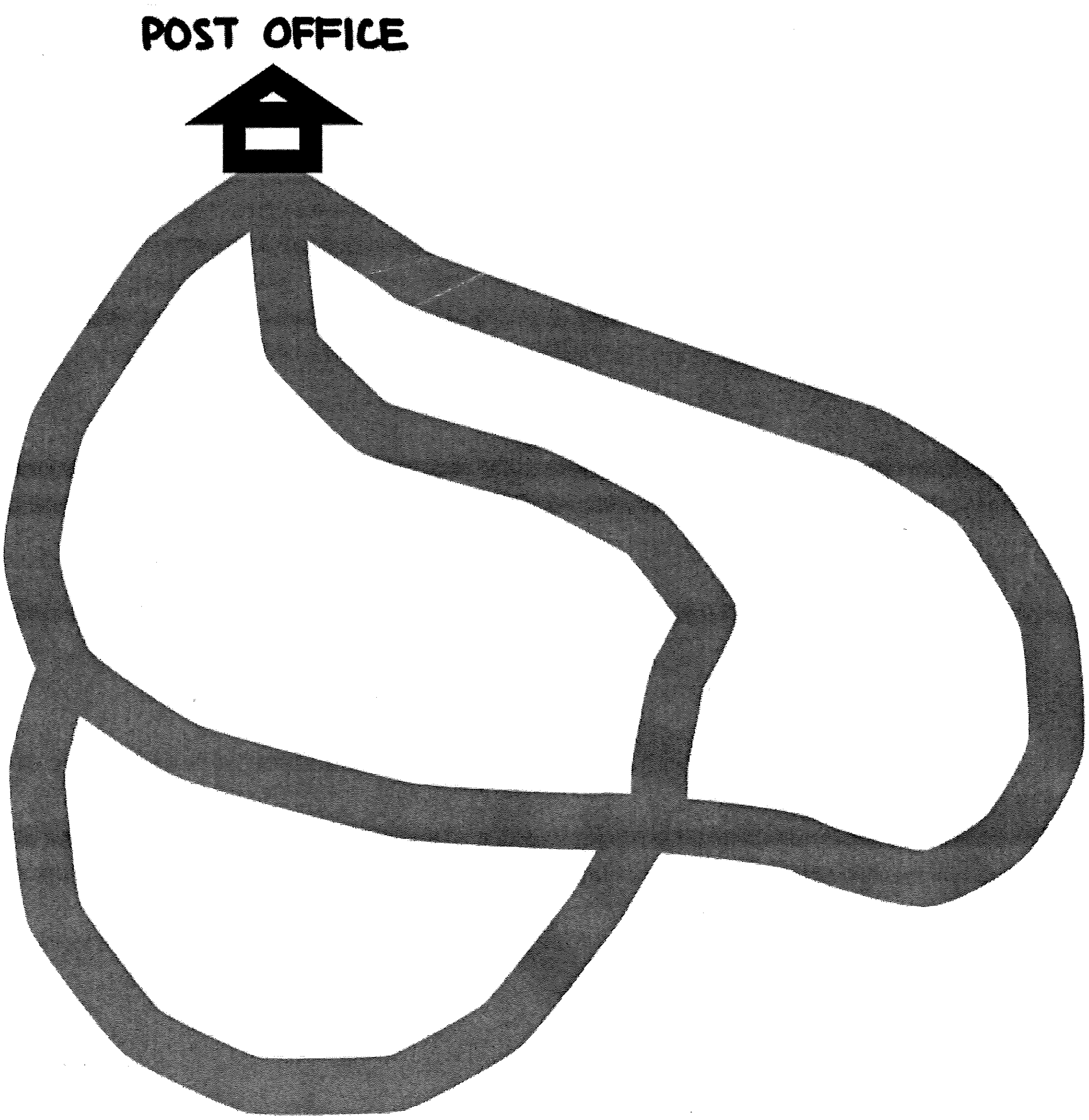
We have reached a contradiction, so it is impossible for everyone to have a unique number of friends. Therefore, there must be at least two people with the same number of friends.

2. Warm Up

Suppose you work at a post office. Your boss has asked you to map out a post carrier route for your town which has mailboxes along the streets in the layout shown below. However, the following requirements must be satisfied:

(1) You must start and end at the post office.
(2) The post carrier must check all the mail boxes.
(3) As gas prices are expensive, the route should not repeat any streets.

Can you find such path? This can't be done as there are vertices with odd degree.



Last quarter, we showed that a statement is logically equivalent to its contrapositive. In this problem, we will use a proof by contrapositive to show that if a graph has an Efficient Cycle, then every one of its vertices has an even degree.

(1) Recall that the contrapositive of the statement

If A then B.

is the statement

If (not B), then (not A).

What is the contrapositive of the statement:

If a graph has an Efficient Cycle, then the degree of every vertex is even.

Use full sentences for your answer.

If there is at least one vertex with odd degree, then the graph does not have an efficient cycle.

(2) If a vertex has an odd degree, why can't every one of its edges belong to a cycle? (Hint: Think about the number of times you enter and exit a vertex.) Explain in full sentences.

A cycle must enter and exit each vertex, so each entry / exit only covers an even number of edges.

(3) Why does this show that if there is a vertex with an odd degree, there are no Efficient cycles? Write your solution down in full sentences.

If a vertex has odd degree, then not every edge belongs to a cycle, which means that it is not efficient as not all edges are covered.

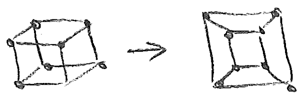
(4) Conclude that if a graph has an Efficient cycle, then all of its vertices have even degrees.

As we have shown the contrapositive, we have shown the original statement, which is equivalent to the contrapositive.

3. Applications of Efficient Paths

Problem 1. A piece of wire is 120 cm long. Can one use it to form the edges of a cube with edges of 10 cm without cutting the wire?

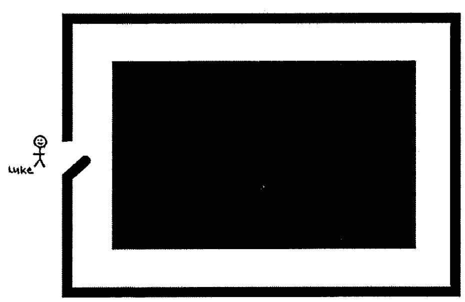
We can "flatten" the cube and draw it into a graph with the corners as vertices:

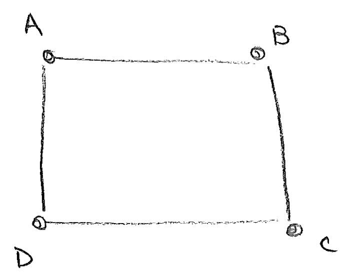


As there are vertices with odd degree, an efficient path doesn't exist, so we can't make a cube out of a single wire.

Problem 2. Let's go back to the original hallway problems one last time. Convert the problem into graphs (you may use your answers from last week) and either write down an efficient cycle or show that an efficient cycle does not exist for the hallways shown below.

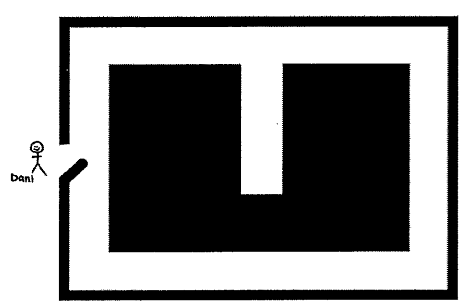
1.

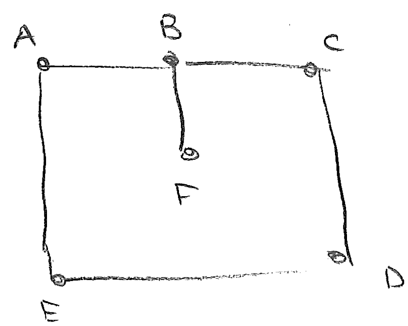




ABCDA.

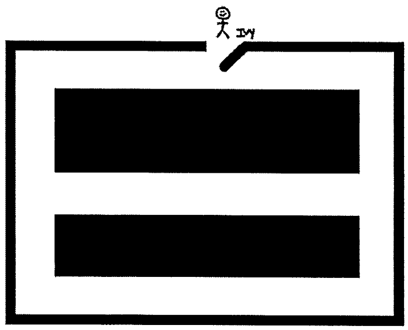
2.

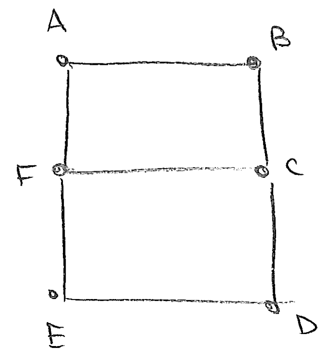




Vertices B and F have odd degree so an efficient path doesn't exist.

3.





Vertices C and F have odd degree so an efficient path doesn't exist.

Problem 3. Given what we currently know, if we had a graph with vertices of all even degrees, would we be able to conclude that an efficient cycle must exist?

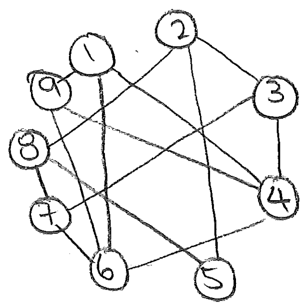
No.

We know that an efficient cycle must have all vertices of even degree, but not the converse.

(For example, "if it's red then the color is warm" is a true statement, but "if the color is warm then it's red" is not true).

4. Problem Solving Using Graphs

Problem 4. Can you write the digits $1-9$ in a row so that the sum of adjacent values are divisible by 5, 7 or 13? For example, $4-9-1$ works, but $1-6-2$ does not. (Hint: Turn the problem in to a graph! What should the vertices be? What about the edges?)



8 5 2 3 7 6 1 4 9

Problem 5. In the country of Numbers, there are cities numbered 1 through 9, and there are highways between every two cities. Each highway is labeled with the two digits of the cities it connects. For example, the freeway from City 3 to City 5 is called "The 35".

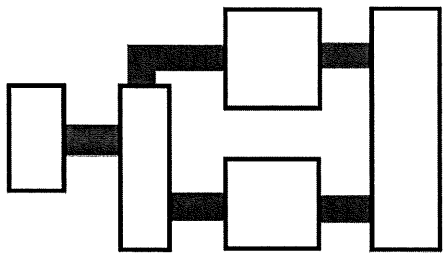
(1) Ivy enjoys the number 3 and will only drive on a highway if the highway number is divisible by 3. Is it possible for her to travel from City 1 to City 7? Explain why or why not in full sentences.

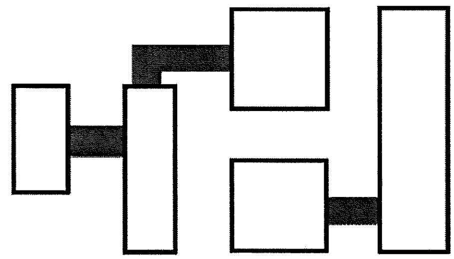
$1 \xrightarrow{\text{The 12}} 2 \xrightarrow{\text{The 27}} 7$

(2) Adrian loves the number 9, and will only drive on a highway if the highway number is divisible by 9. Is it possible for him to travel from City 1 to City 9? Explain why or why not in full sentences.

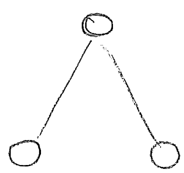
He cannot, as there are no highways leading to city 9 which are divisible by 9.

Problem 6. Dani is designing a new set of corridors for Math Science which are to be safe in the case of an earthquake. Each corridor connects two rooms together. Every room must be connected by a series of corridors and rooms, but the number of corridors that one has to travel through to get to any other room isn't so important. Here is an example of a corridor system that would work, and one that would not:

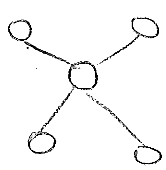


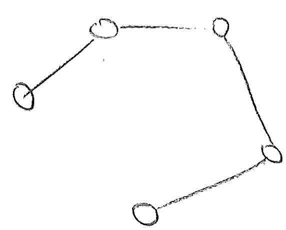


(1) Due to budget cuts, Dani is ordered to build a corridor system for three rooms that uses the smallest number of corridors. How would she do this?

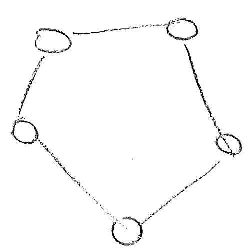


(2) Dani is ordered to build a corridor system for five rooms that uses the smallest number of corridors. Can you come up with two different floor plans that work?

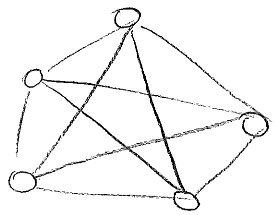




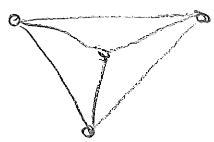
(3) Dani wants to design a corridor system for five rooms that is still usable if a single corridor collapses. How many corridors does she need to build?



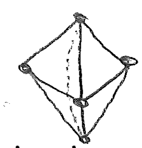
(4) Due to a market boom in the corridor building business, Dani is instructed to make every room connected to every other room with a single corridor in a building with five rooms. How many corridors will Dani have to construct?



(5) Can Dani build the corridor system described in the previous question without having two corridors cross each other? Explain your reasoning in full sentences. Not on a single floor. Once we have 4 vertices, one will be boxed in:



However, in 3 dimensions, we can:



Problem 7. Last week, we showed that the sum of the degrees of vertices in a graph was twice the number of edges.

(1) Describe briefly why this is true.

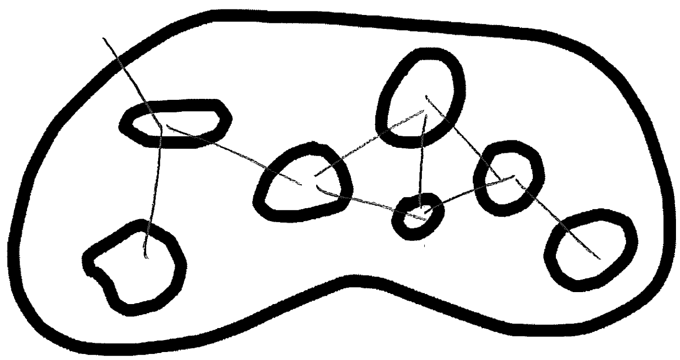
Each edge connects to 2 vertices so it adds 2 to the total degree.

(2) Using the previous problem, conclude that every graph has an even number of vertices with odd degree.

If there were an odd number of vertices with odd degree, then the sum of the degrees would be odd – which is impossible as the sum must be even.

(3) On the first floor of the Math Science building, there is a giant lake with 7 islands in it. Every island has 1, 3 or 5 bridges, and the bridges may connect to the shore or between islands.

(a) Can you put bridges in the following picture so it satisfies the above conditions?



(b) Is there a way of placing the bridges so that they satisfy the above conditions, and no bridge connects to the shore? Explain why or why not in full sentences.

No. Each vertex has odd degree, so an odd number of islands (vertices) with odd degree would not work.

(4) Using the above as inspiration, prove that the number of people who have ever lived on earth and who have shaken hands an odd number of times in their lives, is even.

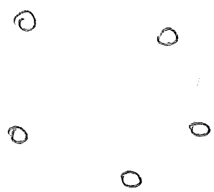
The number of handshakes and people can be represented as a graph. If the number of people who have shaken an odd number of times in their lives is odd, then the sum of all shakes is odd which corresponds to an odd degree sum. This is impossible because the sum of the vertex degrees is twice the number of edges, so it has to be even.

Problem 8. Previously, we used triangles numbers to show that

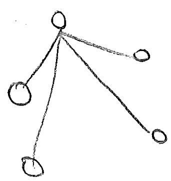
$$1+2+3+\ldots+n = \frac{n\times(n+1)}{2}.$$

This problem will show another proof of the above using graphs.

(1) Draw a graph with 5 vertices.



(2) Select a vertex and then draw edges from that vertex to every other vertex. How many edges have you drawn?



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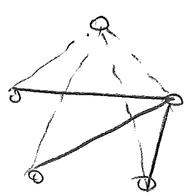
(3) Select another vertex and then draw edges from that vertex to every other vertex. How many *new* edges have you drawn?



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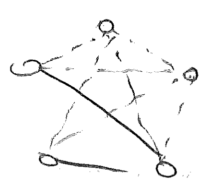
(4) Select a third vertex and then draw edges from that vertex to every other vertex. How many *new* edges have you drawn?



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(5) Repeating this process, how many edges do you end up drawing? (Represent this as a sum of the number of edges you drew on each step.)

$4+3+2+1=10$

(6) Now suppose we have a graph with $n+1$ vertices. How many edges do you draw for the first vertex? The second? The third? ... The $n-1$th?

$n \quad n-1 \quad n-2 \quad \ldots \quad 1$

(7) Repeating this process, how many edges do you end up drawing? (Represent this as a sum of the number of edges you drew on each step.)

$1+2+\ldots+(n-1)+(n-2)+n$

(8) As every vertex has the same degree, what is the sum of the degrees of the vertices?

There are $n+1$ vertices and each has degree n. So $n \times (n+1)$.

(9) How does this show the result that we want? Answer in full sentences.

The vertex sum is twice the number of edges, so the total number of edges is $\frac{n \times (n+1)}{2}$.

We have also shown from (7) that the number of edges is $1+2+\ldots+(n-1)+n$

So $1+2+\ldots+(n-1)+n = \frac{n \times (n+1)}{2}$