

## Tying Up Loose Ends

**The Fundamental Theorem of Arithmetic.** *Any natural number different from 1 can be uniquely represented as a product of primes in increasing order.*

If  $a$  and  $b$  are positive integers, then we say that  $b$  **divides**  $a$  and also that  $a$  **is divisible by**  $b$  if there is a number  $q$  such that  $a = qb$ . The number  $q$  is the **quotient** of  $a$  by  $b$ . To write “ $b$  divides  $a$ ” with symbols, we write  $b \mid a$ . When  $b$  does not divide  $a$ , we write  $b \nmid a$ .

**Examples.**  $4 \mid 12$ , and  $13 \mid 91$ , but  $12 \nmid 25$ .

1. Prove that the product of five consecutive natural numbers is:
  - (a) ... divisible by 30.
  - (b) ... divisible by 120.
  
2. Check whether the number 24681357975318642 is divisible by:  
(a) 2   (b) 4   (c) 5   (d) 10   (e) 3   (f) 9   (g) 11   (h) 7
  
3. An agent at an incompetent intelligence agency was using a code where each number is assigned a distinct letter of the alphabet. The agent writes  $AB \times CD = EFFF$ . Prove that the agent is wrong.
  
4. Find the smallest whole number that when divided by 5, 7, 9, and 11 gives remainders of 1, 2, 3, and 4 respectively.

5. The number 4 has 3 factors—1, 2, and 4. How many factors do each of the following numbers have?

- $35 = 5 \cdot 7$
- $121 = 11^2$
- $n = p^2$ , where  $p$  is a prime number
- $2 = 2$
- $6 = 2 \cdot 3$
- $30 = 2 \cdot 3 \cdot 5$
- $210 = 2 \cdot 3 \cdot 5 \cdot 7$
- $n$ , where  $n$  is the product of the first  $k$  distinct prime numbers
- $10 = 2^1 \cdot 5$
- $20 = 2^2 \cdot 5$
- $40 = 2^3 \cdot 5$
- $80 = 2^4 \cdot 5$
- $n = p^e \cdot q$ , where  $p$  and  $q$  are different prime numbers and  $e$  is a positive integer.
- $100 = 2^2 \cdot 5^2$
- $n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_k^{e_k}$ , where  $p_1, \dots, p_k$  are distinct prime numbers and each  $e_i$  is a positive integer. (Hint: the answer depends on the  $e_i$ 's, but not on the  $p_i$ 's!)

6. Prove that  $k^2 + k - 2$  is never a prime number when  $k$  is a positive integer. (Hint: Calculate the value of the expression for  $k = 1, 2, 3, 4, 5$ . Try to factor each one into a product of two numbers. Can you always factor in such a way?)

7. Positive whole numbers greater than 1 which are *not* prime are called *composite numbers*. The first few are 4, 6, 8, 9, 10, 12, 14, 15, 16, . . .
- (a) What is the longest sequence of consecutive composite numbers less than 100?
  - (b) Suppose you wanted to find 99 consecutive composite numbers. Could you do it? (Hint: Think about the number  $100! = 1 \times 2 \times 3 \times \cdots \times 99 \times 100$ . It's divisible by every number from 1 to 100. What about numbers close to  $100!$ , for instance  $100! + 2$ ?)
8. If I have a clock face balanced flat on top of a pole, I could put equal weights on the numbers 12 and 6, and the clock would stay balanced because 12 and 6 are opposite each other. I could also put 12 weights on the clock, one on each number, and still the clock would remain balanced, since the weights are distributed symmetrically. So I can cover exactly 2 numbers with weights, or I can cover all 12 numbers with weights. What other numbers of weights can I put on the numbers of the clock face and still keep it balanced (and how)?

## Primes of the form $4k + 3$

We can split the odd primes into two distinct groups: those of the form  $4k + 1$  (the first few being 5, 13, 17, 29, 37, ...), and those of the form  $4k + 3$  (the first few being 3, 7, 11, 19, 23, ...). Since we know there are infinitely many primes (and only one even prime!), at least one of these two groups must be infinite. Using an argument similar to Euclid's proof for all primes, we can show that there are infinitely many primes of the form  $4k + 3$ . (In fact there are also infinitely many of the form  $4k + 1$ , but this is more difficult to show.)

1. Show that if you multiply together any two numbers of the form  $4k + 1$  (for instance,  $4m + 1$  and  $4n + 1$ ), you get another number of that form.
2. Explain why any number of the form  $4k + 3$  has a prime factor of the same form.
3. Now we want to show that there are infinitely many primes of the form  $4k + 3$ . Like in Euclid's proof, we will start by assuming the *opposite*: namely, that there are only finitely many primes of this form—and from this try to reach some impossible conclusion, a contradiction.

So assume there are only finitely many primes of the form  $4k + 3$ —call them  $p_1, p_2, \dots, p_r$ . Then consider the number  $N = 4p_1p_2 \cdots p_r + 3$ . What can you say about this number? In particular, which if any of the  $p_i$ 's is it divisible by?

4. Actually, the number  $N$  as defined above *is* divisible by one of the  $p_i$ 's. Change the definition of  $N$  slightly so it's still of the form  $4k + 3$ , but is no longer divisible by any of the primes  $p_i$  of this form. Why is this a contradiction?

## Eggs and Lemonade

1. You and your friend have a large jug holding 8 quarts of lemonade, which you wish to share evenly, 4 quarts each. You also have small and medium-sized jugs, which hold 3 quarts and 5 quarts respectively. Can you measure exactly 4 quarts using only these jugs? If so, how? If not, why not?
2. Now you and your friend have a 10-quart jug of lemonade, and also a 4-quart jug and a 6-quart jug. Can you use these jugs to measure 5 quarts? If so, how? If not, why not?
3. The eggs of the rare and beautiful Quarter-Hour Chicken are delicious, provided they are cooked for precisely 15 minutes. One fine day, you buy such an egg from the local market and return home to cook it for breakfast. But to your shock and horror, you discover that the batteries in your kitchen timer have run out! The only other timepieces you have on hand are two hourglasses—well, not really hourglasses, more like a 7-minute-glass, and an 11-minute-glass. Using these two, how can you measure an interval of exactly 15 minutes?