Take Away Games II: Nim

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The Rules of Nim

- The game of Nim is a two player game.
- There are piles of chips which the players take turns taking chips from.
- During a single turn, a player can only take chips from one pile.
- A player must take at least one chip per turn. However, a player can take more if they choose.

Game 0: One Pile, Last Chip Loses

To check if you've fully grasped the rules, we'll play two easy games. Make a single pile of as many chips as you would like.

1. If the player who takes the last piece *loses*, who will win the game? The first player or the second player?

2. If the player who takes the last piece *wins*, who will win the game? The first player or the second player?

Game 1: Two Equal Piles, Last Chip Loses

For Game 1, create two equal piles with two or more chips in each pile. The player who takes the last chip loses. Play the game a number of times with a partner and figure out the winning strategy.

1. If both players play the game perfectly, will the first player or second player win every time?

2. In order to win, what strategy does the winning player use?

3. Does the strategy still work if you start with any sized piles so long as they are equal? Play more games if necessary.

4. Edge Case: what if each pile only has one chip? Who will win the game? Will the same strategy still work?

Game 2: Two Unequal Piles, Last Chip Loses

For Game 2, create two piles with the first pile having one more chip than the second (i.e. piles of 4 and 3 or piles of 5 and 4).

1. If both players play the game perfectly, will the first player or second player win every time?

2. In order to win, what strategy does the winning player use?

3. Would it would still be a winning strategy if one pile was larger than other by any amount greater than one?

And The Winner Is...

1. Say Player A always goes first and Player B always goes second. In Game 1, you started off with two equal piles and figured out who would win. In Game 2, you started off with two unequal piles and figured out who would win. Can the same player win both games?

2. If I tell you the game state (e.g. the current size of both piles) and whose turn it is, could you tell me who would win? Let's see.

   (a) The pile sizes are 2 and 2. It is Player A's turn. Which player will win?

   (b) The pile sizes are 29 and 27. It is Player A's turn. Which player will win?

   (c) The pile sizes are n and n. it is Player A's turn. Which player will win?

   (d) The pile sizes are n and $n + 97$. It is Player A's turn. Which player will win?

3. What game state is always a winning state for the player taking their turn?

4. What game state is always a losing state for the player taking their turn?

Game 3: Three Piles, Last Chip Loses

For Game 3, create three piles. Two piles must to have the same number of chips; the third pile can have any number of chips.

1. If both players play the game perfectly, will the first player or second player win every time?

2. In order to win, what strategy does the winning player use?

3. Who wins if the third pile has the same number of chips as the first two piles?

4. Does it matter how many chips are in the third pile? What about the winning strategy makes this the case?

Nim Mania!

1. If you started a game with as many piles as you like and each pile had one chip, in what scenario would the first player win? What scenarios would the second player win?

2. In a three pile game where the player who takes the last chip losses, there are 13 losing states. Prove the state with pile sizes of (1, 2, 3) is a losing state for a Player A who is about to take their turn. Hint: show that all six moves the player can take results in a winning position for Player B based on what you've learned from Games 1-3. (Showing something is true for all finite cases is called "proof by exhaustion" in mathematics.)

3. In Game 1, you determined who would win if the game started with with two equally-sized piles. Who would win if there were any even number of equally-sized piles? Hint: break the piles into pairs and treat each pair as its own two-pile game.