FINISHING UP SUCCESSIVE DIFFERENCES

BEGINNER CIRCLE 4/17/2016

Problem 1. Let S be the sequence of square numbers. Prove that $dS = O$, where O is the sequence of odd numbers.

	S_1	S_2	S_3	S_4	S_5		S_n	S_{n+1}
S:	1	4	9	16	25	...	n^2	$(n+1)^2$

$dS_n = S_{n+1} - S_n = (n+1)^2 - n^2$

$= n^2 + 2n + 1 - n^2$

$= 2n+1$ = the sequence of odd numbers

Problem 2. We've noticed that with the triangular numbers T, that $dddT = 0$, and with the square numbers, $dddS = 0$ (Where 0 means the sequence of all zeroes.) Prove that the sequence of square numbers S have the property that

$$d^3 = dddS = 0.$$

As shown above, $dS_n = 2n+1$.

So $ddS_n = dS_{n+1} - dS_n = [2(n+1)+1] - (2n+1) = 2n+2+1-2n-1 = 2$

So $dddS_n = ddS_{n+1} - ddS_n = 2 - 2 = 0$

// NOTE: In order for this to be a proper "proof" we cannot just write out some terms of S and call it a day, as that would only prove that $dddS_n = 0$ for the first couple of terms. We <u>must</u> use a generic expression of $dddS_n$ using n's! :)

Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics.

Problem 3. Can you find a (non-zero) sequence such that $dF = F$?

We can rewrite the question as $dF_n = F_n$.

We know that $dF_n = F_{n+1} - F_n$

and we want that to be equal to F_n.

So we get : $dF_n = \boxed{F_{n+1} - F_n = F_n}$

Rewriting the boxed equation gives us $F_{n+1} = 2F_n$.

This means that to get the next digit in F, we have to multiply our current number by two.

So one answer could be $F = 1, 2, 4, 8, 16, \dots$ (answers vary)

Problem 4. Can you find a (non-zero) sequence where $ddF = F$?

Since $dF = F$ in our above sequence,

$ddF = d(\underbrace{dF}_{F}) = dF = F$

So the same sequence works.

Problem 5. Find a (non-zero) sequence that has the property that

$dF = F$ with all the numbers shifted to the right by one place

Let's call "F with all the numbers shifted to the right" : F^*

So if $F = \overset{F_1}{1}, \overset{F_2}{2}, \overset{F_3}{3}, \dots$

Then $F^* = \underset{F_1^*}{\_\_}, \underset{F_2^*}{1}, \underset{F_3^*}{2}, 3, \dots$

Notice that $F_2^* = F_1$, $F_3^* = F_2, \dots$ So $F_n^* = F_{n-1}$

So our original question can be written as

$dF_n = F_n^* = F_{n-1}$

So using the same method as above,

$dF_n = \boxed{F_{n+1} - F_n = F_{n-1}}$

$F_{n+1} = F_n + F_{n-1} \Rightarrow$ "To get the next number, add current number to the one before it"

One example is Fibonacci sequence : $F = 0, 1, 1, 2, 3, 5, 8, 13, \dots$

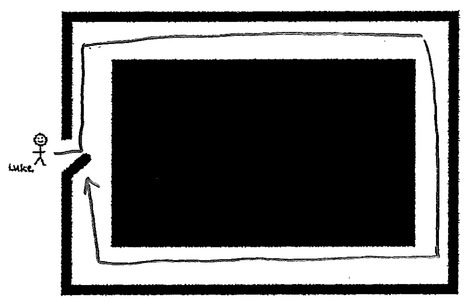
HANDSHAKING AND CHASING KIDS

BEGINNER CIRCLE 11/4/2012

1. Kids running around Math Science

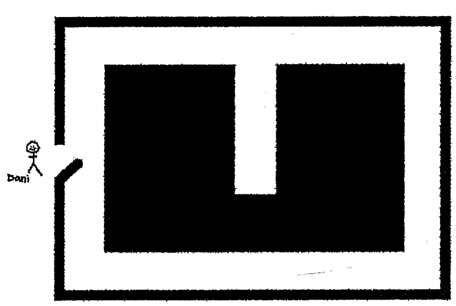
At the end of every day, Luke, Dani, Ivy, Sam and Adrian have to check the hallways of Math Science to look for runaway students. As they are all lazy, they only want to walk down each corridor once as they look for students. So they take paths which are efficient, where they walk down each corridor just once, and start and stop at the same spot. On which floors do the instructors get to be lazy?

Problem 1. Luke always checks the 1st floor. Is it possible for him to check all the corridors without walking any corridor twice?



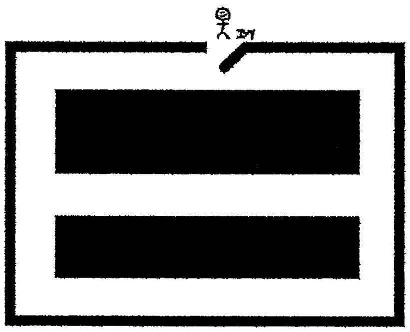
Yes

Problem 2. Dani always checks the second floor. Can you find a path that goes through every corridor exactly once? Why or why not?



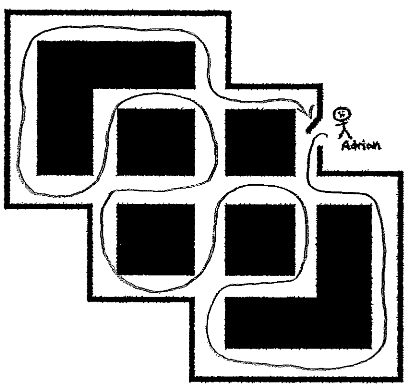
NO.

Problem 3. When Math Circle finishes, Ivy runs down to the fifth floor (which, confusingly is the ground floor). Can Ivy check all of the corridors without checking any corridor twice?



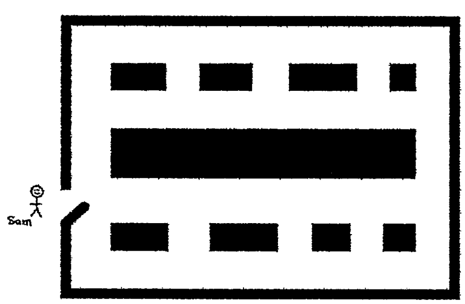
No.

Problem 4. As Adrian is the fastest of all of the Math Circle instructors, he checks the largest floor (which happens to be floor seven). Can he check every corridor without going to any corridor twice?



Yes.

Problem 5. Sam decides to check the confusing eighth floor of Math Science. Can he do it efficiently?



No.

2. GRAPHS

A **graph** is a collection of objects called **vertices** and a collection of **edges** that go in between them, which have the following properties:

- There is at most one edge connecting any two vertices
- Every edge connects two different vertices

Here are some examples of graphs and non-graphs

FIGURE 2.1. Things that are Graphs

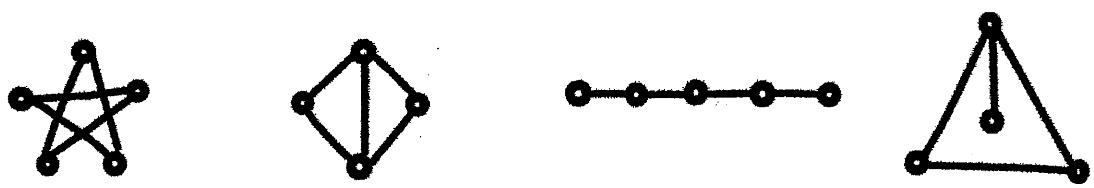
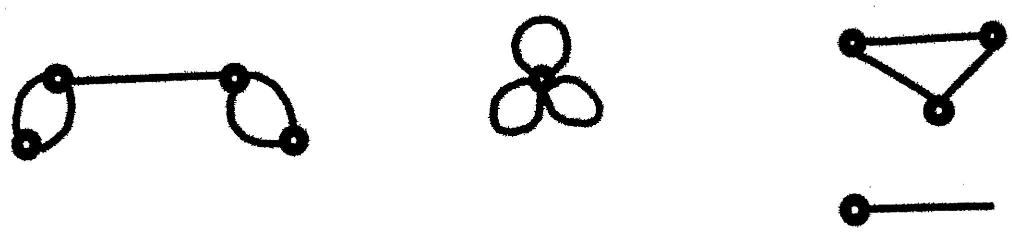
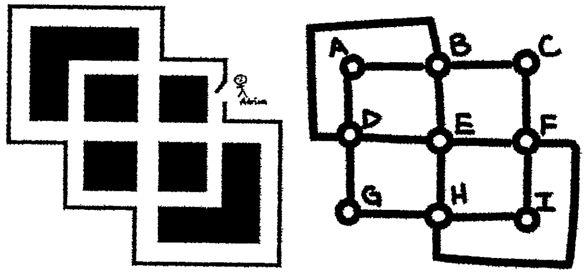


FIGURE 2.2. Things that are not Graphs



When we have a graph, one of the properties we are most interested in is the number of edges that are connected to each vertex. If v is a vertex of a graph, then the **degree** of v (written $\deg v$) is the number of edges that are connected to that vertex. If no edges are connected to the vertex, we say it has degree 0.

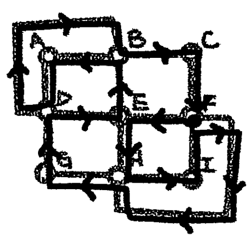
Let us return to the problem with checking for students. These problems can be represented with graphs!
We can represent the places where the corridors intersect or end as vertices and represent the corridors themselves as edges. For example, the problem with Adrian's corridor looks like this:



If we want to talk about a specific corridor in this graph, we can just say the two vertices that sit on the ends of that corridor. For instance, suppose Adrian is standing next to the corridor CF. It is important to notice that we could just as well say that Adrian is sitting next to the corridor FC. If we want to talk about a path that Adrian takes, we can just specify the order of the vertices that he visits. For example, the sequence

$$FEHIFHGDEBADBCF$$

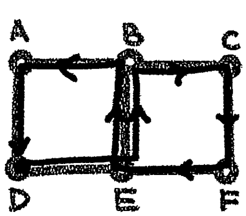
represents the following path:



Furthermore, on the graph below, the sequence

$$DEBCFEBAD$$

represents the following path:

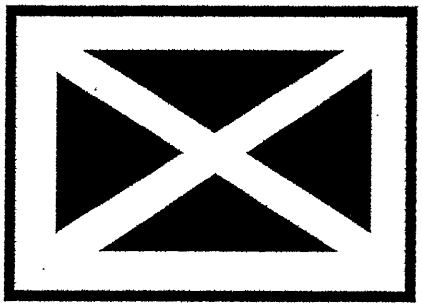


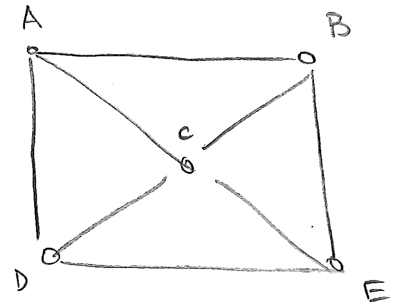
So a **path** is a sequence of vertices that satisfies the following property:

(1) Every adjacent pair of vertices in the sequence are connected by an edge.

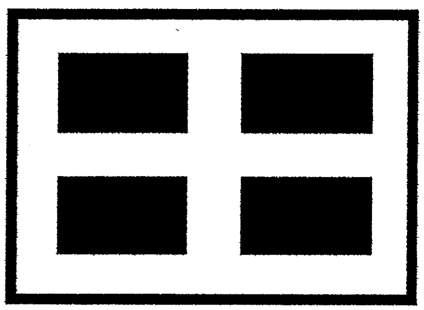
Problem 6. Can you convert the following hallways into graphs? Label the vertices with letters.

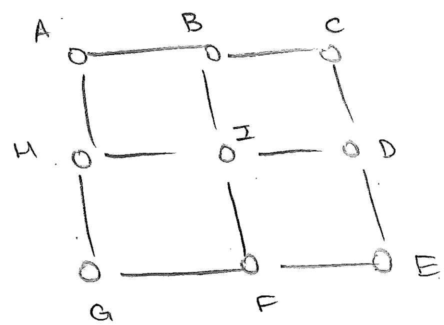
1.



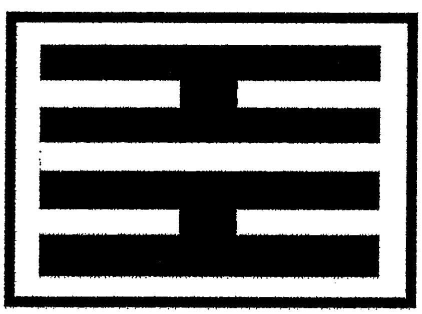


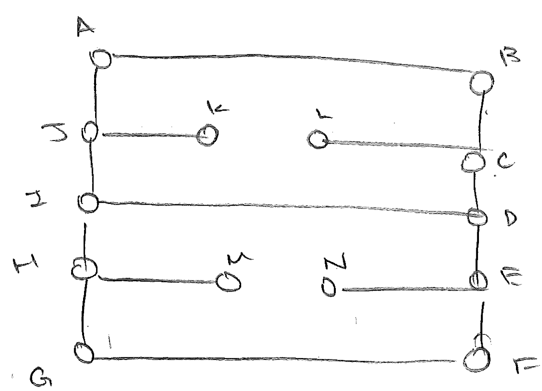
2.





3.



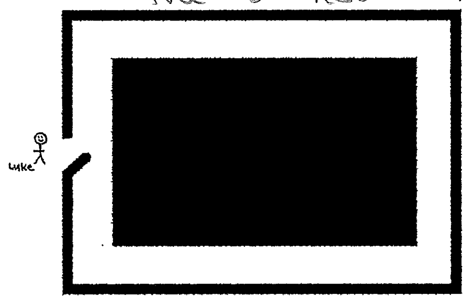


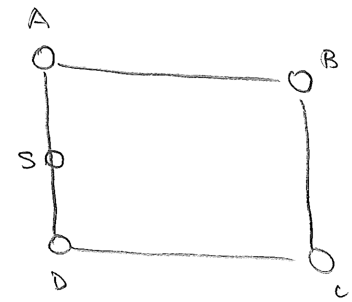
Note: these answers may vary. Any node that has degree 2 can be removed/added arbitrarily.

Problem 7. Go back to the original hallway problems. Convert the problem into graphs, and then write down the paths using a sequence of vertices.

Note: we can add a node "S" to represent the starting position without affecting the correctness. Answers without the "S" node are okay too.

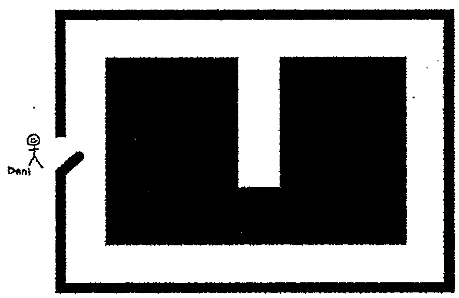
1.

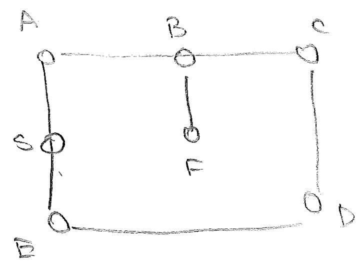




S A B C D S

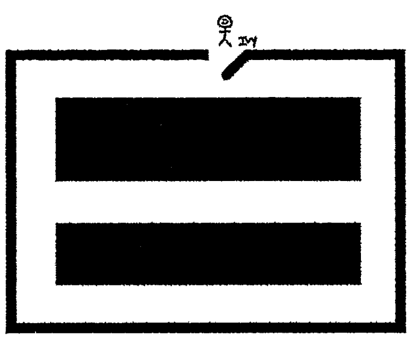
2.

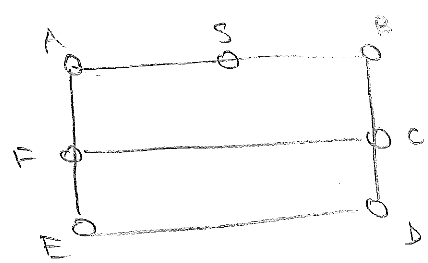




SA B F B C D E S

3.

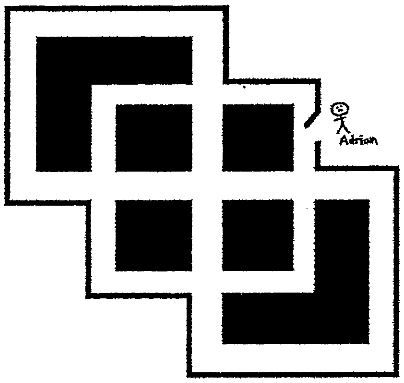


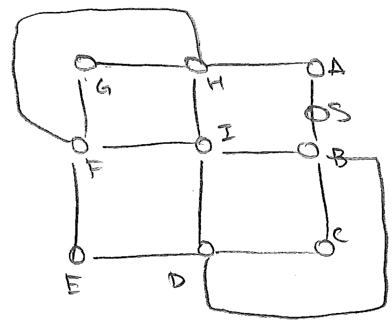


SBCD E F C F A S

Note: paths do not have to be "efficient"

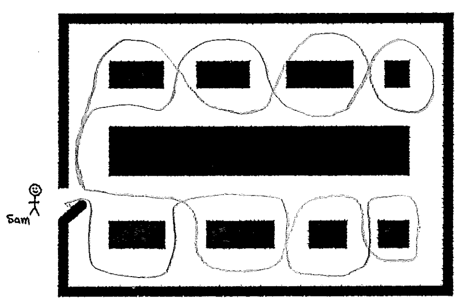
4.

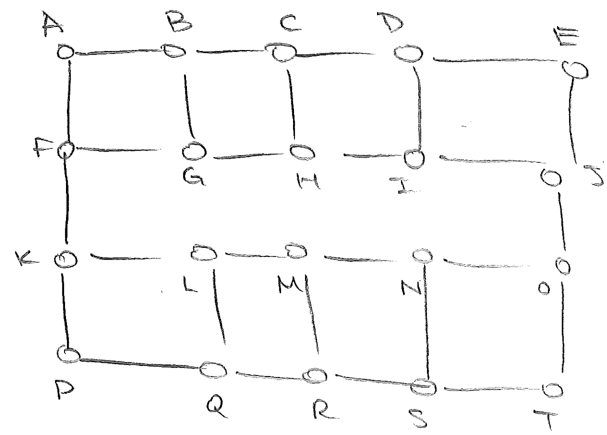




S B D C B I D E F I H G F H A S

5.

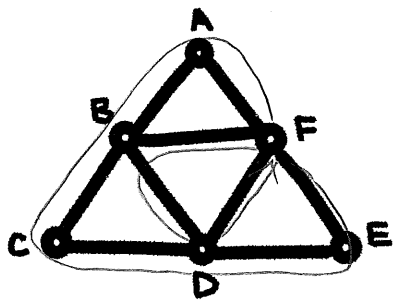




K P Q L M R S N O T S N M R Q L K F A B G H I D E J I D C H G B A F K

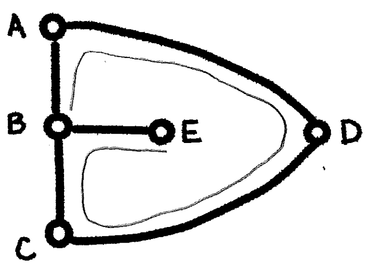
Problem 8. Can you find a path that visits every edge only once on the graphs below? Write out the vertices that the path visits in order. It is ok if your path visits the same vertex multiple times.

1.



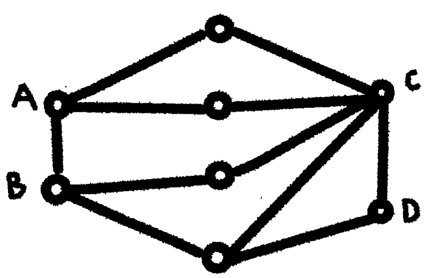
FABCDEFB DF

2.



EBCD AB

Problem 9. Find the degrees of the following vertices in this graph:



(1) What is $\deg A$? 3

(2) What is $\deg B$? 3

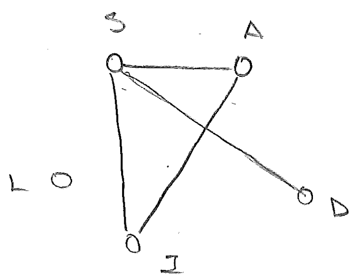
(3) What is $\deg C$? 5

(4) What is $\deg D$? 2

What is the sum of the degrees of this graph? 22

Problem 10. [Handshake lemma] Luke is having a party, and invites over his friends: Sam, Adrian, Dani, and Ivy. At the party, people shake hands many times. Luke has an obsession with hand shaking, and he wants to know exactly how many handshakes have happened. He learns that Sam shook hands with everybody besides Luke. In addition to these handshakes, Adrian and Ivy shook hands as well. Luke shook no hands, as he was so busy counting handshakes.

(1) Can you turn this problem into a graph? (Hint: Use the people as the vertices)



(2) How many handshakes occurred that evening?

4 handshakes (same # as edges).

(3) Luke goes around and asks each person how many hands they shook, and then sums up those numbers. How many handshakes does Luke count this way?

8 handshakes.

(4) Using the above as an inspiration, explain using complete sentences why the sum of the degrees of vertices in a graph is always twice the number of edges.

Each edge connects two vertices so the edge adds two degrees to the total number of degrees of vertices in the graph.

(5) Conclude that the sum of the degrees of vertices is always even.

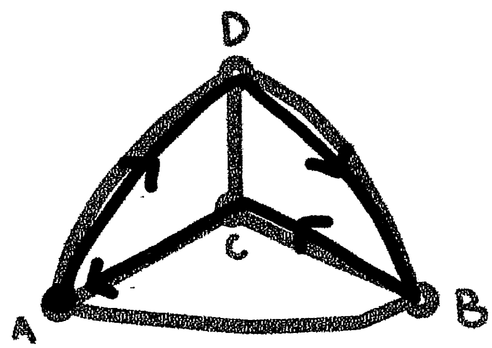
Since the sum of the degrees is twice the number of edges, the sum is always even.

3. Cycles

A **cycle** is a path that starts and ends at the same point. For example, the path

$$ADBCA$$

is a cycle in the graph

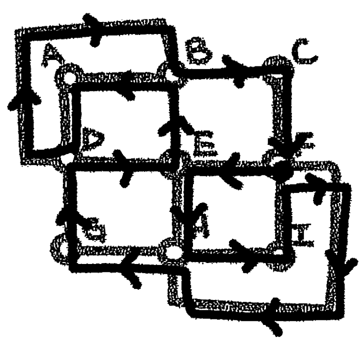


A cycle is called **Efficient** if it visits every edge but does not repeat any edges.

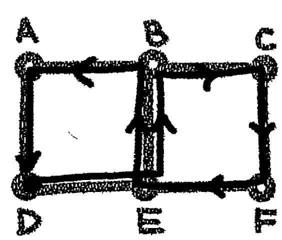
For example, the sequence

$$FEHIFHGDEBADBCF$$

is an example of an efficient cycle on the following graph:



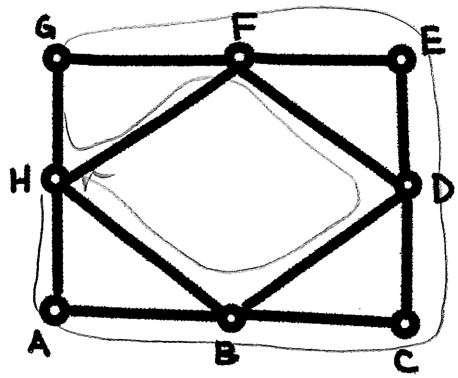
However, the following path is not an efficient cycle:



This is because the edge BE is repeated in the sequence $DEBCFEBAD$.

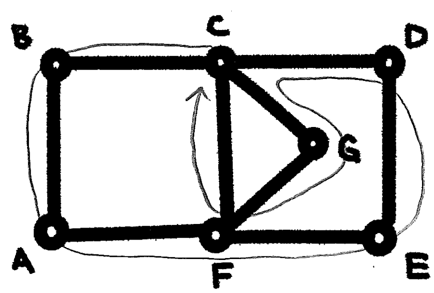
Problem 11. Can you find an Efficient cycle for the following graphs? Write down the cycle.

1.



HABCDEFGHFDBH

2.



CBAFEDCGFC

Problem 12. Last quarter, we showed that a statement is logically equivalent to its contrapositive. In this problem, we will use a proof by contrapositive to show that if a graph has an Efficient Cycle, then every one of its vertices has an even degree.

(1) Recall that the contrapositive of the statement

If A then B.

is the statement

If (not B), then (not A).

What is the contrapositive of the statement:

If a graph has an Efficient Cycle, then the degree of every vertex is even.

Use full sentences for your answer.

If there is at least one vertex with an odd degree, then the graph does not have an efficient cycle.
Note that answers with "every vertex is odd" are wrong!

(2) If a vertex has an odd degree, why can't every one of its edges belong to a cycle? (Hint: Think about the number of times you enter and exit a vertex.) Explain in full sentences.

A cycle must enter and exit each vertex. So each entry / exit only covers an even number of vertices

(3) Why does this show that if there is a vertex with an odd degree, there are no Efficient cycles? Write your solution down in full sentences.

If a vertex has odd degree, then not every edge belongs to a cycle, which means it is not an efficient cycle as not all the edges are covered.

(4) Conclude that if a graph has an Efficient cycle, then all of its vertices have even degrees.

As we have proved the contrapositive, we have proved the original statement.

Last quarter, we also did some proofs by contradiction. In a proof by contradiction, if we wanted to prove a statement, we first looked at its opposite. Then we showed that its opposite statement proved something impossible (for example, that $0 = 1$).

Adrian has an obsession with sorting things. He is having a party next week, and decides that he will rank the 50 guests by popularity. He will say that one person is more popular if they have more friends at the party. However, he finds that this is impossible, no matter who he invites and how many friends they have. This is because there are always at least two people with the same number of friends. To help him out, we will show that what he is trying to do is impossible by proving the following statement:

There must be at least two people with the same number of friends.

(1) For a proof by contradiction, we first assume the opposite statement is true. What is the opposite of the statement:
There are two people at the party with the same number of friends at the party?
Use full sentences for your answer.

Each person has a unique (different) number of friends at the party.

(2) Can you turn this problem into one about graphs? What are the vertices? What about the edges?

Each person is represented by a vertex and a "friendship" is represented by an edge.

(3) Let X be the number of friends (at the party) a person at the party has. What are the possible values of X?

$X = 0$ to 49.

(someone at the party can have no friends at the party, be friends with everyone at the party, or somewhere in between).

(4) Why does this tell us that there is a person who has 49 friends? Why does this also show that someone at the party has no friends? Explain in full sentences.

Since X can take on 50 values and there are 50 people with a different number of friends each, each of the numbers from 0 to 49 must be assigned to a person at the party. So someone is assigned 49 and someone else is assigned 0.

(5) Why is it that the person with 49 friends is friends with everybody in the room?

There are 50 people in the room, including the person with 49 friends, so that person is friends with everyone (except themselves).

(6) Why is it impossible for someone to be friends with everybody in the room?

We also said that there was someone with no friends at the party, so it's impossible for the person with 49 friends to be friends with everyone (as they are not friends with the person with 0 friends)

(7) Conclude that there are two people that have the same number of friends.

Since we have reached a contradiction, it is impossible for everyone to have a unique number of friends. Therefore, there must be at least two people with the same number of friends.