

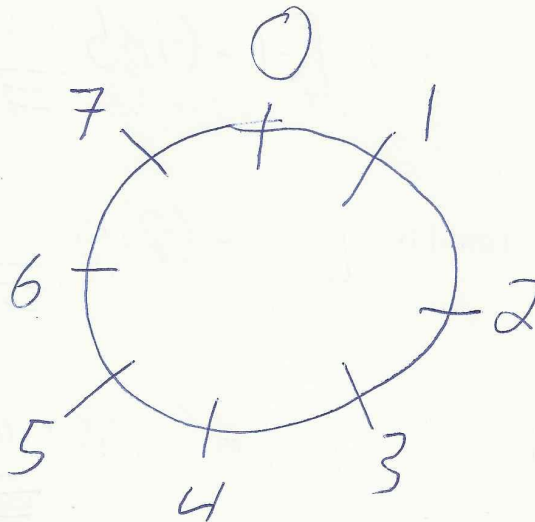
Modular Arithmetic, Part II

April 10, 2016

Modular arithmetic. Last week, we introduced *modular congruencies*. Two whole numbers a and b are said to be *congruent modulo n* , often written $a \equiv b \pmod{n}$, if they give the same remainders when divided by n .

Warm Up

Draw a picture showing mod 8 arithmetic on a circle (similar to mod 12 arithmetic on a clock).



(1) Reduce the numbers in modular arithmetic.

$$(a) 17 \equiv 2 \pmod{5} \quad [17 = 3 \times 5 + \underline{\underline{2}}]$$

$$(b) 29 \equiv 9 \pmod{10} \quad [29 = 2 \times 10 + \underline{\underline{9}}]$$

$$(c) 433551 \equiv 1 \pmod{2} \quad [433551 = 2 \times 216775 + \underline{\underline{1}}]$$

$$(d) 91 \equiv 0 \pmod{13} \quad [91 = 7 \times 13 + \underline{\underline{0}}]$$

$$(e) -1 \equiv 4 \pmod{5} \quad [-1 = (-1) \times 5 + \underline{\underline{4}}]$$

$$(f) -10 \equiv 2 \pmod{6} \quad [-10 = (-2) \times 6 + \underline{\underline{2}}]$$

$$(g) -1 \equiv n-1 \pmod{n} \quad [-1 = (-1) \times n + \underline{\underline{(n-1)}}]$$

(2) Fill in the addition and multiplication tables.

+	EVEN	ODD
EVEN	Even	Odd
ODD	Odd	Even

×	EVEN	ODD
EVEN	Even	Even
ODD	Even	Odd

(3) Fill in the addition and multiplication tables below in the case of modulus $n = 2$.

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

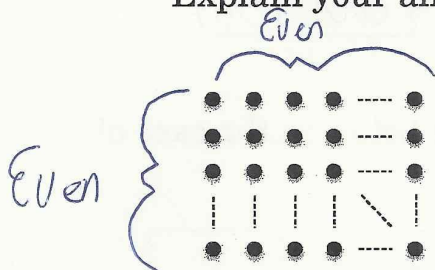
(a) Any even number can be represented by $2 \times n$. Reduce this number in mod 2 arithmetic.

$$2 \times n \equiv 0 \pmod{2} \quad [2 \times n = 2 \times n + \underline{\underline{0}}]$$

(b) Any odd number can be represented by $2 \times n + 1$. Reduce this number in mod 2 arithmetic.

$$2 \times n + 1 \equiv 1 \pmod{2} \quad [2 \times n + 1 = 2 \times n + \underline{\underline{1}}]$$

- (4) There are 931 students in Franklin Elementary School. During an assembly, all of the students gather together and form a grid. The figure below shows what the grid looks like. The number of students in each row is even, and the number of students in each column is even. Did all of the students attend the assembly? Explain your answer.



Not all the students attended.
The grid fits an even number of students, but an odd number of students showed up.

- (5) Use modular arithmetic to solve the following problems.

- (a) Winter went to the store to buy candy, juice, and ice cream. Candy costs \$2.00, juice costs \$4.00, and ice cream costs \$6.00. She spent \$23.00. Is this possible? Explain why or why not.

Not possible. Any combination of items will result in an even total. Winter spent an odd amount of dollars however.

$$[\text{any combination of items}] \pmod{2} \neq 23 \pmod{2}$$

- (b) Lucy went to the store to buy milk, bread, eggs, and pencils. Milk costs \$4.42, bread costs \$1.64, pencils cost \$0.72, and eggs cost \$4.98. Her total bill ended up being \$31.63. Is this possible? Explain why or why not.

Not possible. Same reasoning as above

$$[\text{any combination of items}] \pmod{2} \neq 3163 \pmod{2}$$

(*hint: use pennies, not \$X.XX)

- (6) Fill in the addition and multiplication tables in the cases of modulus $n = 4$, $n = 5$, and $n = 6$.

(a) Modulo 4:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

\times	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(b) Modulo 5:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

\times	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(c) Modulo 6:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(7) Use the addition tables you filled out in the previous problems to solve the following subtraction problems.

$$3 - 2 \equiv 1 \pmod{4}$$

$$4 - 3 \equiv 1 \pmod{5}$$

$$4 - 1 \equiv 3 \pmod{5}$$

$$5 - 2 \equiv 3 \pmod{6}$$

(8) Use the multiplication tables to solve the following division problems

$$3 \div 1 \equiv 3 \pmod{4}$$

$$3 \div 2 \equiv 4 \pmod{5}$$

$$4 \div 3 \equiv 3 \pmod{5}$$

$$2 \div 3 \equiv 4 \pmod{5}$$

$$5 \div 1 \equiv 5 \pmod{6}$$

(9) Zero Divisors

(a) A non-zero number k is called a *zero divisor* when there is another number l such that $k \times l$ is equivalent to zero in arithmetic mod n .

(i) Can you explain the name "zero divisor"?

A divisor is a number that divides a number

A zero divisor is a number that divides zero in particular.

(ii) Are there zero divisors in usual arithmetic?

No.

(iii) Use the multiplication tables you filled out earlier to find all zero divisors in the following cases:

• Modulus: $n = 4$; Zero divisors: 2

$2 \times 2 = 0 \pmod{4}$ so 2 is a zero divisor

• Modulus: $n = 5$; Zero divisors: none

• Modulus: $n = 6$; Zero divisors: 2, 3

$2 \times 3 = 0 \pmod{6}$

(b) Without constructing any more multiplication tables, find all zero divisors for the following values of n :

(i) $n = 7$

none

(ii) $n = 9$

3

(iii) $n = 11$

none

(iv) $n = 12$

2, 3, 4, 6

[Credit: Lily]

The zero divisor for any n will be all of its factors other than 1 and n

(c) What can you say about zero divisors if n is prime?

There will be no zero divisors if n is prime.

What is Congruency?

You're all familiar with with equality ($=$), but modular arithmetic uses congruency (\equiv). Why does modular arithmetic use congruency? Take for example $11 \equiv 7 \pmod{4} = 3$. I am saying "11 is congruent to $7 \pmod{4}$ and $7 \pmod{4}$ equals 3." 11 is **congruent** to $7 \pmod{4}$ because $11 \pmod{4}$ **equals** $7 \pmod{4}$. I say $7 \pmod{4}$ **equals** 3 because the result of $7 \pmod{4}$ is actually 3. 11 doesn't equal 3, so 11 cannot be **equal** to $7 \pmod{4}$, it can only be **congruent**.

- (10) Provide numbers that are both congruent and equal to the following. I want you to be creative, so the two numbers you provide for each answer cannot be the same (e.g. $3 \equiv 7 \pmod{4} = 3$ even though it is true).

(a) $9 \equiv 2 \pmod{7} = 2$

(b) $11 \equiv 16 \pmod{5} = 1$

(c) $35 \equiv 23 \pmod{12} = 11$

(d) $-1 \equiv 7 \pmod{8} = 7$

(e) $24 \equiv (20 \pmod{4}) = 0$

* All congruent values can be found by adding ~~the~~ n to whatever you're taking the mod of.
Ex:

$$2+7 = 9 \equiv 2 \pmod{7} =$$

$$2-7 = -5 \equiv 2 \pmod{7}$$

- (11) Make up your own problem about modular arithmetic. Show the problem to your teacher and then to a partner.

* Use any problem from this or the previous worksheet as a model.