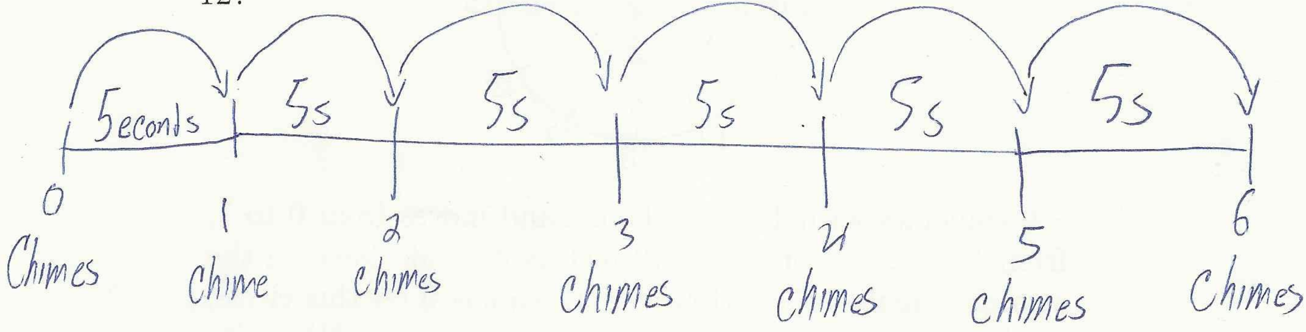
LAMC Junior Circle April 3, 2016

Problem 1. It takes a grandfather's clock 30 seconds to chime 6 o'clock. Assuming that the time of each chime is negligible compared to the time intervals between the chimes, how much time would it take the clock to chime 12?



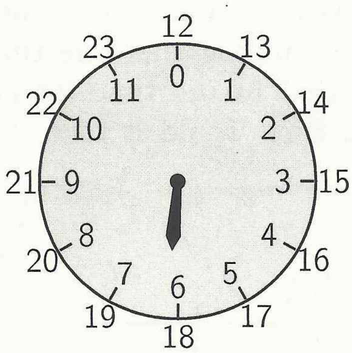
* It takes 5 seconds per chime.

$$5\,\frac{\text{seconds}}{\text{chime}} \cdot 12 \text{ chimes} = 60 \text{ seconds.}$$

It would take 60 seconds to chime 12 o'clock

Clock Arithmetic or a Circle as a Number Line

One way to turn a circle into a number line is to divide it into twelve equal parts. In this case, one step is usually called one hour.



0 coincides with 12. The hour hand moves from 0 to 1, from 1 to 2, ... from 11 to 12 just as it would have on the straight number line. However, 12 equals 0 on this circle, so there it goes again, from 1 to 2, and so on. We write down the fact that 12 equals 0 as

$$12 \equiv 0 \ (mod\ 12) \tag{0.1}$$

and read it as *12 is congruent to 0 modulo 12.* The usual "=" sign is reserved for the straight number line; we use "≡" on the circle instead. The *mod* 12 symbol tells us that the circle is divided into 12 equal parts, so 12 coinsides with 0, 13 – with 1, 14 – with 2, and so on. Or in the new notations,

$13 \equiv 1 \ (mod\ 12),\ 14 \equiv 2 \ (mod\ 12), \ldots, 23 \equiv 11 \ (mod\ 12),$

$$24 \equiv 12 \equiv 0 \ (mod\ 12).$$

Problem 2. Divide the following numbers by 12, and write the remainders of the divisions.

$15 \div 12 =$ 1 R 3 (because $15 = (12 \times \underline{1}) + \underline{3}$

$21 \div 12 =$ 1 R 9

$37 \div 12 =$ 3 R 1

$46 \div 12 =$ 3 R 10

$80 \div 12 =$ 6 R 8

* Notice the Relationships

Now, write down the modular congruencies for the following numbers.

$15\ (mod\ 12) \equiv$ 3

$21\ (mod\ 12) \equiv$ 9

$37\ (mod\ 12) \equiv$ 1

$46\ (mod\ 12) \equiv$ 10

$80\ (mod\ 12) \equiv$ 8

These are remainders after division.

Problem 3. Write down the modular congruencies for the following additions.

$$9 + 4 \equiv 1 \quad (mod\ 12)$$

$$18 + 8 \equiv 2 \quad (mod\ 12)$$

*add the numbers (9+4=13), then take the mod (13 mod 12 = 1 mod 12

When subtracting two numbers 'a' and 'b', finding a - b means finding a number 'c' such that c + b = a.
For example, 5 - 3 =2 because when you add 2 to 3, you get 5.

The same occurs with modular arithmetic.

Write down the modular congruencies for the following subtractions.

$$8 - 3 \equiv 5 \quad (mod\ 12)$$

$$1 - 11 \equiv 2 \quad (mod\ 12)$$

$$4 - 15 \equiv 1 \quad (mod\ 12)$$

* Subtract the numbers (8-3=5), then take the mod (5 mod 12 = 5 mod 12)

* To find the mod of negative numbers (like -10 mod 12), keep adding the mod base until you have a positive number.

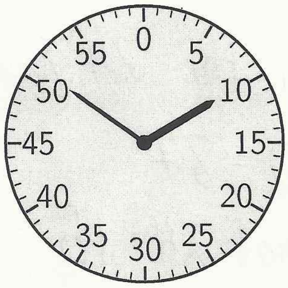
Example:

$(\underline{-10})(mod\ \underline{12}) = (\underline{-10+12})(mod\ 12) = \underline{2}\ mod\ 12$

add together

The number is positive, so stop

Another standard way to turn a circle into a number line is to divide it into 60 equal parts. Depending on the situation, the unit step is called either a minute or a second.



All the numbers living on this number line are considered modulo 60. In particular, $60 \equiv 0$ (*mod* 60). There are 60 minutes in an hour.

Problem 4.

$72 \equiv$ 12 $(mod\ 60)$

$135 \equiv$ 15 $(mod\ 60)$

$55 + 55 \equiv$ 0 $(mod\ 60)$

$-15 \equiv$ 45 $(mod\ 60)$

$240 - 59 \equiv$ 1 $(mod\ 60)$

Problem 5. An experiment in a biological lab starts at 7:00 AM and runs for 80 hours. What time will it end?

* Hint: Use 24 hour clock (where 1pm = 13)

7am = 7 on the 24 hour clock

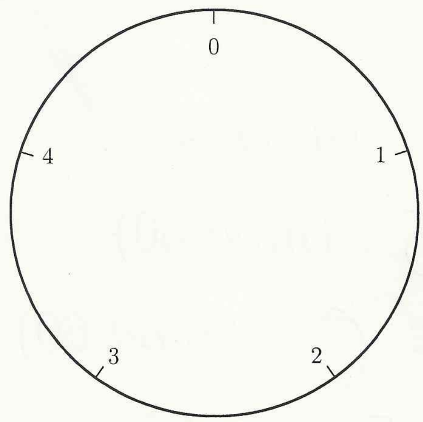
$7 + 80 = 87$

$87 \bmod 24 = 15$

15 = 3pm on 12-hour clock.

$\boxed{\text{The experiment will end at 3pm}}$

In the following problems, we will consider the *mod* 5 arithmetic, that of a circle divided into five equal parts.



Problem 6. Write down the modular congruencies for the following multiplation problems.

$$2 \times 3 \equiv \quad 1 \quad (mod\ 5)$$

$$4 \times 4 \equiv \quad 1 \quad (mod\ 5)$$

$$5 \times 7 \equiv \quad 0 \quad (mod\ 5)$$

* Multiply first and then take the mod of that number
Ex: $2 \times 3 = 6$ then $6 \bmod 5 = 1$

Similar to how subtraction is related to addition, division is related to multiplcation. Finding $a \div b$ means finding a number 'c' such that
$b \times c = a$.

For example, $6 \div 3 = 2$ because $3 \times 2 = 6$.

Write down the modular congruencies for the following division problems.

$$1 \div 2 \equiv \quad 3 \quad (mod\ 5)$$ (Because $3 \times 2 = 6$ and $6 \bmod 5 = 1$)

$$1 \div 3 \equiv \quad 2 \quad (mod\ 5)$$

$$1 \div 4 \equiv \quad 4 \quad (mod\ 5)$$ (Becouse $4 \times 4 = 16$ and $16 \bmod 5 = 1$)

Problem 7. Write down the modular congruencies using "mod (*your age*)".

$11 + 39 \equiv 4 \pmod{23}$

$22 + 17 \equiv 16 \pmod{23}$

$33 - 11 \equiv 22 \pmod{23}$

$2 - 50 \equiv 21 \pmod{23}$

$1 \times 10 \equiv 10 \pmod{23}$

$4 \times 4 \equiv 16 \pmod{23}$

	mod 8	mod 9	mod 10	mod 11	mod 12
11 + 39	2	5	0	6	2
22 + 17	7	3	9	6	3
33 − 11	6	4	2	0	10
2 − 50	0	6	2	7	0
1 × 10	2	1	0	10	10
4 × 4	0	7	6	5	4