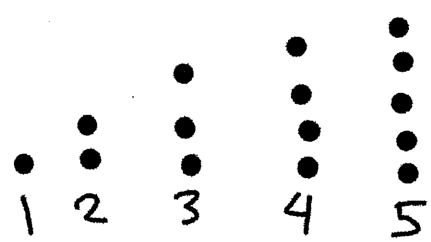
(SOLUTIONS)

SUCCESSIVE DIFFERENCES

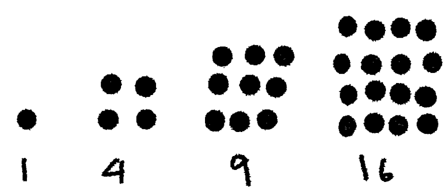
BEGINNER CIRCLE 4/10/2016

1. Shape Numbers

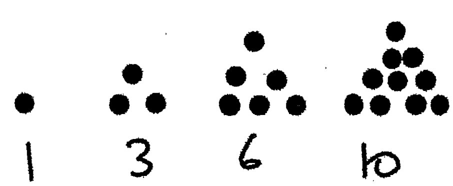
We all know about numbers. But what about the numbers that arise from looking at different shapes. For instance we have line numbers:



which are easy to compute – the nth line number is just n. We also have the square numbers:

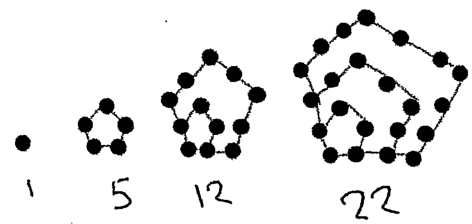


which are also fairly easy to compute: the nth square number is just n^2. But what about triangular numbers?



Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics .

or pentagonal numbers?

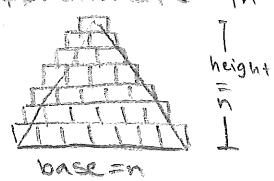


and so on?

Problem 1. Using the formula for the area of a triangle, find an approximate formula for triangular numbers. Can you make this approximation better?

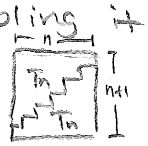
Let T_n be the n^{th} triangle number.

We can approximate T_n by using the area of a triangle:



$T_n \approx \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}n^2$

We can do this better by looking at T_n a different way and doubling it:



$2T_n = n \times (n+1)$ so $T_n = \frac{n \times (n+1)}{2}$

Problem 2. Write each triangular number as the sum of smaller line numbers. **Explain why your formula works.**

We add the next line number $(n+1)$ to get to the next triangle number so:

$T_1 = 1$

$T_2 = 1 + 2$

$T_3 = 1 + 2 + 3$

$\vdots$

$T_n = 1 + 2 + \dots + n$.

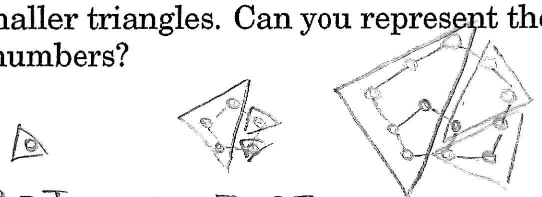
Problem 3. Using the previous problem as inspiration, think about what is the difference between the 4th triangular number and the 5th triangular number? What about the 5th and the 6th triangular numbers? What about the nth and $(n+1)$th? Why?

$T_5 - T_4 = (1+2+3+4+5) - (1+2+3+4) = 5$

$T_6 - T_5 = (1+2+3+4+5+6) - (1+2+3+4+5) = 6$

$T_{n+1} - T_n = (1+2+\dots+n+n+1) - (1+2+\dots+n) = n+1.$

Problem 4. Take each pentagon in pentagonal numbers and break it down into several smaller triangles. Can you represent the pentagonal numbers as a sum of triangular numbers?



$P_1 = T_1$ $P_2 = T_2 + 2T_1$ $P_3 = T_3 + 2T_2$ ...

$P_n = T_n + 2T_{n-1}$

Problem 5. What is the difference between the 4th and 5th pentagonal numbers? What about the 5th and 6th? Can you find a pattern? Explain in full sentences

$P_5 - P_4 = 13$

$P_6 - P_5 = 16.$

$P_n - P_{n-1} = 3n-2$

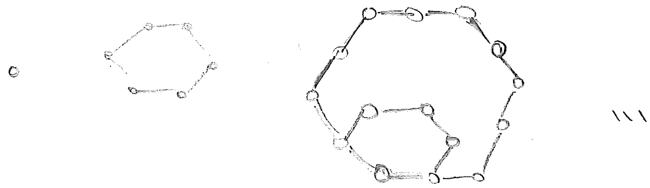
We can break each pentagonal number into 3 triangle numbers, so getting to the next pentagonal number is equivalent to adding the next layers onto the 3 triangle numbers. So to get from P_{n+1} to P_n, we add $n + (n-1) + (n-1)$ $= 3n-2$.

Problem 6. Using your pattern in the problem above, find all of the pentagonal numbers up to the 8th pentagonal number.

Pentagonal numbers: 1 5 12 22 35 51 70 91

How much to add to get to the next one: +4 +7 +10 +13 +16 +19 +21

Problem 7. Draw a few pictures of what hexagonal numbers should look like. Use the same methods as in the previous three problems to find all hexagonal numbers up to the 6th hexagonal number.

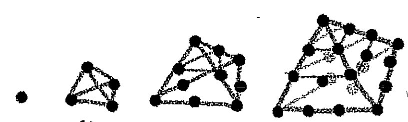


hexagonal # : 1 6 15 28 45 66

difference : +5 +9 +13 +17 +21

(4n−3)

Problem 8. Tetrahedral numbers are given by little pyramids with triangular bases. Can you write each tetrahedral number as a sum of triangular numbers? Write an explanation in full sentences.



1 4 10 20

$Tet_n = T_1 + T_2 + \dots + T_n$.

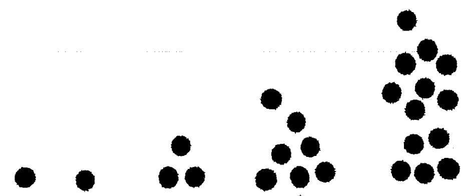
Each tetrahedral number is obtained by adding the next triangle number to the base of the tetrahedral number.

Problem 9. Using the problem above as motivation, find all tetrahedral numbers up to the 7th one.

Tetrahedral # : 1 4 10 20 35 56 84

Difference : +3 +6 +10 +15 +21 +28

Problem 10. Gooey Kablooie numbers are given by the pictures below. Use the techniques that you have developed in the previous 5 problems to figure out what the 8th Gooey Kablooie number is.



1 1 3 7 13

We find the difference between each number first :

0 2 4 6 ...

So the difference between the n^{th} # and the one before it is given by $2n-2$.

Gooey Kablooie #: 1 1 3 7 13 21 31 43

Difference: +0 +2 +4 +6 +8 +10 +12

2. Successive Differences

In the warm-up , we looked at sequences of numbers and tried to figure out the pattern to the sequence. Sometimes, it is really easy for us to figure out the pattern: for example, we all know how to figure out the next number in the pattern:

$$2, 4, 6, 8, \ldots$$

In fact, we can do even better than just figuring out the next number in the pattern, we can provide a formula that computes the nth number, where $n = 1, 2, 3, 4...$

$$2n$$

One way to identify a pattern is to look at the differences between successive numbers in the pattern. Let's give some language to describe these patterns.

Definition 1. A **sequence** is a bunch of numbers that come in a particular order. When we talk about the whole sequence, we use a capital letter. For example, the sequence of even numbers might be written as

$$E = 2, 4, 6, 8$$

When we want to refer to a specific element in the sequence, we will use lower case letters. For example, if we wish to refer to the first, fifth or nth numbers in the sequences, we will donate them as

$$e_1 = 2$$
$$e_5 = 10$$
$$e_n = 2n$$

Definition 2. If we are given a sequence

$$A = a_1, a_2, a_3, \ldots$$

we define the **difference sequence** of A (which we will denote as dA) to be the sequence of differences between the elements of A. For example, if the sequence A is

$$A = 2, 3, 5, 8, 9, 9, 9, 9, \ldots$$

then

$$dA = 1, 2, 3, 1, 0, 0, 0, \ldots$$

Formally, the sequence dA is given by entries

$$b_i = a_{i+1} - a_i$$

where $i = 1, 2, 3, 4...$

Problem 11. Let N be the sequence of numbers,

$$N = 1, 2, 3, 4, \ldots$$

What is dN? What is b_3?

$dN = (2-1), (3-2), (4-3), \ldots$

$= 1, 1, 1, \ldots$

$b_3 = 1$

Problem 12. Let S be the sequence of square numbers,

$$S = 1, 4, 9, 16, \ldots$$

What is dS? What is b_5?

$dS = (4-1), (9-4), (16-9), \ldots$

$= 3, 5, 7, \ldots$

$b_5 = 11$

Problem 13. Find three sequences A_1, A_2 and A_3 that have the property

$$dA = 0, 0, 0, 0, \ldots$$

$A_1 = 1, 1, \ldots$

$A_2 = 2, 2, \ldots$

$A_3 = 1/2, 1/2, \ldots$

(Anything with a repeating constant number works)

Problem 14. Frequently, we need to take the difference of a sequence several times. Let T be the sequence of triangular numbers (from the warm-up.) What is dT? What about ddT?

$T = 1, 3, 6, 10, \ldots$

$dT = 2, 3, 4, \ldots$

$ddT = 1, 1, 1, \ldots$

Problem 15. Let A be a sequence. Suppose we know that $a_1 = 0$, and we know that $dA = T$, where T is the sequence of triangular numbers. Can you find a_5, the fifth entry of the sequence A? Can you give a name to sequence A?

$A = 0, 1, 4, 10, 20, \ldots$

$a_5 = 20$

Tetrahedral Numbers

Problem 16. Let S be the sequence of square numbers. Prove, without simply writing out dS, that $dS = O$, where O is the sequence of odd numbers. (Hint: write the formula for dS.)

Let S_n be the n^{th} square number. ($S_n = n^2$)

Each entry of dS is given by:

$dS_n = S_{n+1} - S_n$

$= (n+1)^2 - n^2$

$= (n^2 + 2n + 1) - n^2$

$= 2n+1$

$dS_n = 2n+1$ gives the sequence of odd numbers, so

$dS = O$