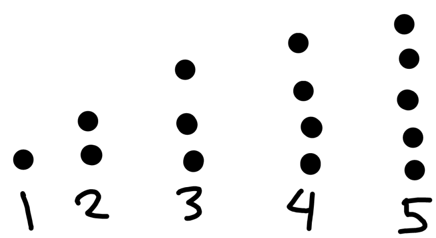
SUCCESSIVE DIFFERENCES

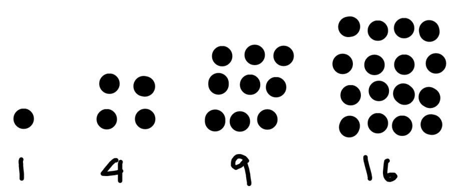
BEGINNER CIRCLE 4/10/2016

1. Shape Numbers

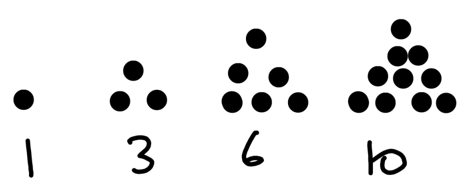
We all know about numbers. But what about the numbers that arise from looking at different shapes. For instance we have line numbers:



which are easy to compute – the nth line number is just n. We also have the square numbers:



which are also fairly easy to compute: the nth square number is just n^2. But what about triangular numbers?



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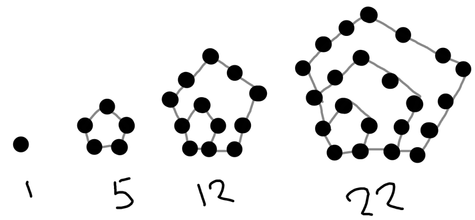
or pentagonal numbers?



and so on?

Problem 1. Using the formula for the area of a triangle, find an approximate formula for triangular numbers. Can you make this approximation better?

Problem 2. Write each triangular number as the sum of smaller line numbers. Explain why your formula works.

Problem 3. Using the previous problem as inspiration, think about what is the difference between the 4th triangular number and the 5th triangular number? What about the 5th and the 6th triangular numbers? What about the nth and $(n + 1)$th? Why?

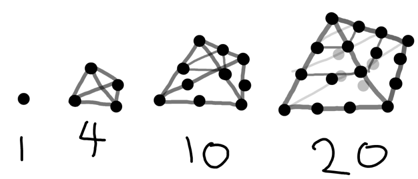
Problem 4. Take each pentagon in pentagonal numbers and break it down into several smaller triangles. Can you represent the pentagonal numbers as a sum of triangular numbers?

Problem 5. What is the difference between the 4th and 5th pentagonal numbers? What about the 5th and 6th? Can you find a pattern? Explain in full sentences

Problem 6. Using your pattern in the problem above, find all of the pentagonal numbers up to the 8th pentagonal number.

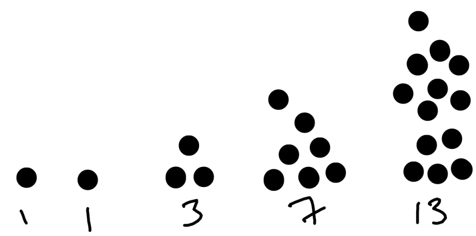
Problem 7. Draw a few pictures of what hexagonal numbers should look like. Use the same methods as in the previous three problems to find all hexagonal numbers up to the 6th hexagonal number.

Problem 8. Tetrahedral numbers are given by little pyramids with triangular bases. Can you write each tetrahedral number as a sum of triangular numbers? Write an explanation in full sentences.



Problem 9. Using the problem above as motivation, find all tetrahedral numbers up to the 7th one.

Problem 10. Gooey Kablooie numbers are given by the pictures below. Use the techniques that you have developed in the previous 5 problems to figure out what the 8th Gooey Kablooie number is.



2. Successive Differences

In the warm-up , we looked at sequences of numbers and tried to figure out the pattern to the sequence. Sometimes, it is really easy for us to figure out the pattern: for example, we all know how to figure out the next number in the pattern:

$$2, 4, 6, 8, \ldots$$

In fact, we can do even better than just figuring out the next number in the pattern, we can provide a formula that computes the nth number, where $n = 1, 2, 3, 4...$

$$2n$$

One way to identify a pattern is to look at the differences between successive numbers in the pattern. Let's give some language to describe these patterns.

Definition 1. A **sequence** is a bunch of numbers that come in a particular order. When we talk about the whole sequence, we use a capital letter. For example, the sequence of even numbers might be written as

$$E = 2, 4, 6, 8$$

When we want to refer to a specific element in the sequence, we will use lower case letters. For example, if we wish to refer to the first, fifth or nth numbers in the sequences, we will donate them as

$$e_1 = 2$$
$$e_5 = 10$$
$$e_n = 2n$$

Definition 2. If we are given a sequence

$$A = a_1, a_2, a_3, \ldots$$

we define the **difference sequence** of A (which we will denote as dA) to be the sequence of differences between the elements of A. For example, if the sequence A is

$$A = 2, 3, 5, 8, 9, 9, 9, 9, \ldots$$

then

$$dA = 1, 2, 3, 1, 0, 0, 0, \ldots$$

Formally, the sequence dA is given by entries

$$b_i = a_{i+1} - a_i$$

where $i = 1, 2, 3, 4...$

Problem 11. Let N be the sequence of numbers,

$$N = 1, 2, 3, 4, \dots$$

What is dN? What is b_3?

Problem 12. Let S be the sequence of square numbers,

$$S = 1, 4, 9, 16, \dots$$

What is dS? What is b_5?

Problem 13. Find three sequences A_1, A_2 and A_3 that have the property

$$dA = 0, 0, 0, 0, \dots$$

Problem 14. Frequently, we need to take the difference of a sequence several times. Let T be the sequence of triangular numbers (from the warm-up.) What is dT? What about ddT?

Problem 15. Let A be a sequence. Suppose we know that $a_1 = 0$, and we know that $dA = T$, where T is the sequence of triangular numbers. Can you find a_5, the fifth entry of the sequence A? Can you give a name to sequence A?

Problem 16. Let S be the sequence of square numbers. Prove, without simply writing out dS, that $dS = O$, where O is the sequence of odd numbers. (Hint: write the formula for dS.)

Problem 17. We've noticed that with triangular numbers T, $dddT = 0$, and with square numbers S, $dddS = 0$, where 0 means the sequence of all zeroes. Show that the sequence of cubic numbers C has the property that

$$d^4C = ddddC = 0$$

Problem 18. Can you find a (non-zero) sequence such that $dF = F$?

Problem 19. Can you find a (non-zero) sequence where $ddF = F$?

Problem 20. Find a (non-zero) sequence that has the property that

$dF = F$ with all the numbers shifted to the right by one place.

3. Polynomials

First off, what is a polynomial?

Definition 3. A **polynomial** $f(n)$ is an equation made by multiplying and adding together numbers and powers some variable n. For instance, the following are all polynomials:

$$n^2 + 2 \qquad 22n + 34n^3 5 \qquad \frac{347928}{2837} \qquad (n^2+1)(n+5)$$

Definition 4. The **degree** of a polynomial $f(n)$ is the highest exponent that appears in the polynomial. For instance

$$n^3 + 2n + 6$$

has degree 3, while the polynomial

$$(n^4 + 2n)(n^2 + 1)$$

has degree 6, because when you expand the polynomial, you will get $n^4 \times n^2 = n^6$.

One convenient way to make sequences is with polynomials. For example we have already seen the sequence of squares, given by

$$s_n = n^2$$

and the sequence of triangular numbers, which is given by

$$t_n = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

Our question is, which sequences can be given by a polynomial?

Problem 21. Find the polynomials that give the sequences of Odd (O), Even (E), and Threeven (T) numbers respectively:

$$O = 1, 3, 5, 7, 9, \ldots$$
$$E = 2, 4, 6, 8, 10, \ldots$$
$$T = 3, 6, 9, 12, \ldots$$

Problem 22. Can you find a polynomial that fits the sequence

$$A = 1, 0, 1, 4, 9, \ldots$$

Problem 23. Can you find a polynomial that fits the sequence

$$A = 1, -1, 1, 15, \ldots$$

As you can see, it is quite hard to find the polynomial that fits the given elements! However, it is easier to find the sequences by looking at differences.

Problem 24. Let A and B be two different sequences. We define the **sum sequence** to be the sequence where each of their elements is summed together. For example of the sequences

$$A = 1, 3, 2, 5, 4, \ldots$$

$$B = 1, 1, 2, 2, 3, \ldots$$

$$A + B = 2, 4, 4, 7, 7, \ldots$$

Can you show that

$$d(A + B) = dA + dB$$