Quadratics

LA Math Circle (Advanced)

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Today we'll be interested in various properties of quadratic (degree two) polynomials. The standard form of such a polynomial is

$$ax^2 + bx + c \tag{1}$$

where a, b, and c are numbers and x is a variable. There isn't much we can say about our quadratic just by looking at it in this form, so our goal will be to use algebraic manipulations to write

$$ax^2 + bx + c = a(x - d)^2 + e \tag{2}$$
$$ax^2 + bx + c = a(x - r_1)(x - r_2) \tag{3}$$

for some numbers d and e and some other (possibly complex) numbers r_1 and r_2. First, let's start with a warmup:

Problem 1 Prove the following (extremely important) formulas:

$(a + b)(c + d) = ac + ad + bc + bd$

$(a + b)(a - b) = a^2 - b^2$

$(a + b)^2 = a^2 + 2ab + b^2$

Problem 2 Prove one more important formula:

$(x - a)(x - b) = x^2 - (a + b)x + ab$

We can use problem 2 to put some especially simple quadratics into the form (3) from page 1. For instance, if our quadratic is

$$x^2 - 4x + 3,$$

if we want to write $x^2 - 4x + 3 = (x - r_1)(x - r_2)$, then by problem 2, all we have to do is find two numbers r_1 and r_2 that add up to 4 and multiply to 3, like 1 and 3! Now it's easy to see that $x^2 - 4x + 3 = (x - 1)(x - 3)$. Also, it's very easy to solve the equation $x^2 - 4x + 3 = 0$ now: there are exactly two solutions, $x = 1$ and $x = 3$.

Problem 3 Factor the following quadratics:

$x^2 - 8x + 16$

$x^2 - 7x + 10$

$x^2 + x - 2$

Problem 4 Solve the following equations:

Hint: $ax^2 + bx + c = a\big(x^2 + \frac{b}{a}x + \frac{c}{a}\big)$

$3x^2 - 3 = 0$

$4x^2 - 8x - 32 = 0$

$2x^2 + 4x + 2 = 0$

So it's pretty easy to solve a quadratic if it happens to have integer roots, but unfortunately, most "real life" quadratics do not. For these, we will use a method called completing the square to convert our quadratic to form (2) from page 1. Here's the idea:

Suppose we start with the quadratic

$$x^2 - 4x + 1,$$

and we want to write it in the form

$$x^2 - 4x + 1 = (x - d)^2 + e.$$

Note that if we expand this equation, we get

$$x^2 - 4x + 1 = (x - d)^2 + e = x^2 - 2dx + d^2 + e.$$

From this we see that $2d = 4$ or $d = 2$. So,

$$x^2 - 4x + 1 = (x-2)^2 + e = x^2 - 4x + 4 + e.$$

But now if we set $e = -3$, then $4 + e = 1$, so

$$(x-2)^2 - 3 = x^2 - 4x + 1,$$

as desired.

Problem 5 Express the following quadratics in form (2):

$x^2 + 2x - 3$

$x^2 - 8x + 26$

Problem 6 Express the following quadratic in form (2), where b and c are arbitrary numbers:

$x^2 + bx + c$

Problem 7 Express the following quadratic in form (2), where a, b, and c are arbitrary numbers and $a \neq 0$:

$ax^2 + bx + c$

Problem 8 Form (2) is very nice, partly because it makes it easy to graph our quadratic, and to find their minimum or maximum value. Sketch a graph of each of the following quadratics. Be sure to include in your sketch the location and value of the maximum or minimum of the function.

$x^2 + 2x - 3$

$-2x^2 + 8x - 2$

Problem 9 A dog owner has 100 feet of fence and wants to fence off a rectangular region in his yard. What is the largest area he can possibly fence off?

Problem 10 Solve the following quadratics by first putting them in form (2):

$x^2 + 2x - 3$

$-2x^2 + 8x - 2$

Problem 11 Solve the general quadratic equation

$$ax^2 + bx + c = 0$$

using your solution to problem 7.

You should have found the two solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is called the quadratic formula, and while most times in math it's better to reason things out, this formula is **worth memorizing**.

Problem 12 A ball is thrown into the air by a $y_0 = 2\ m$ tall person with initial speed $v_0 = 20\ m/s$. The height y of the ball is known to satisfy

$$y = y_0 + v_0 t + \frac{1}{2} g t^2$$

where $g = 9.8\ m/s^2$ is the acceleration of gravity and t is the elapsed time in seconds. At what time does the ball hit the ground?

Problem 13 Given a quadratic $ax^2 + bx + c$, define the discriminant of to be $\Delta = b^2 - 4ac$. Use the quadratic formula to prove the following about the equation $ax^2 + bx + c = 0$:

The equation has exactly two real solutions if and only if $\Delta > 0$.

The equation has exactly one real solution if and only if $\Delta = 0$.

The equation has no real solution if and only if $\Delta < 0$.

Problem 14 Prove that if $ax^2 + bx + c \geq 0$ for every real number x. Prove that $\Delta \leq 0$ (a picture and half a sentence would suffice).

Problem 15 Apply the result from problem 14 to the quadratic:

$$(a_1 - xb_1)^2 + (a_2 - xb_2)^2 + \cdots + (a_n - xb_n)^2.$$

Problem 16 By the quadratic formula, the equation

$$x^2 + bx + c = 0$$

has solutions

$$r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2},\ r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Prove that $r_1 + r_2 = -b$ and $r_1 r_2 = c$.