Peculiar Primes

Remember: a positive number is *prime* if it is only divisible by two numbers—itself and 1. The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, \ldots$.

Every integer greater than 1 can be written in a *unique* way as a product of prime numbers. For instance, $100 = 2 \cdot 2 \cdot 5 \cdot 5$, $23 = 23$ (since it's already prime), and $91 = 7 \cdot 13$.

1. Prove that among any three consecutive odd numbers, one of them is divisible by 3.

2. Write 1,000,000,000 as the product of two numbers, neither of which has any zeroes when written.

3. Find the smallest whole number that when divided by 5, 7, 9, and 11 gives remainders of 1, 2, 3, and 4 respectively.

4. The number 4 has 3 factors—1, 2, and 4. How many factors do each of the following numbers have?

   - $35 = 5 \cdot 7$
   - $121 = 11^2$
   - $n = p^2$, where p is a prime number
   - $2 = 2$
   - $6 = 2 \cdot 3$
   - $30 = 2 \cdot 3 \cdot 5$
   - $210 = 2 \cdot 3 \cdot 5 \cdot 7$
   - n, where n is the product of the first k distinct prime numbers
   - $10 = 2^1 \cdot 5$

- $20 = 2^2 \cdot 5$
- $40 = 2^3 \cdot 5$
- $80 = 2^4 \cdot 5$
- $n = p^e \cdot q$, where p and q are different prime numbers and e is a positive integer.
- $100 = 2^2 \cdot 5^2$
- $n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_k^{e_k}$, where $p_1, \ldots, p_k$ are distinct prime numbers and each e_i is a positive integer. (Hint: the answer depends on the e_i's, but not on the p_i's!)

5. Prove that $k^2 + k - 2$ is never a prime number when k is a positive integer. (Hint: Calculate the value of the expression for $k = 1, 2, 3, 4, 5$. Try to factor each one into a product of two numbers. Can you always factor in such a way?)

6. Positive whole numbers greater than 1 which are *not* prime are called *composite numbers*. The first few are $4, 6, 8, 9, 10, 12, 14, 15, 16, \ldots$.

   (a) What is the longest sequence of consecutive composite numbers less than 100?

   (b) Suppose you wanted to find 99 consecutive composite numbers. Could you do it? (Hint: Think about the number $100! = 1 \times 2 \times 3 \times \cdots \times 99 \times 100$. It's divisible by every number from 1 to 100. What about numbers close to 100!, for instance $100! + 2$?)

7. If I have a clock face balanced flat on top of a pole, I could put equal weights on the numbers 12 and 6, and the clock would stay balanced because 12 and 6 are opposite each other. I could also put 12 weights on the clock, one on each number, and still the clock would remain balanced, since the weights are distributed symmetrically. So I can cover exactly 2 numbers with weights, or I can cover all 12 numbers with weights. What other numbers of weights can I put on the numbers of the clock face and still keep it balanced (and how)?