GAMES

BEGINNERS CIRCLE 04/03/2016

1. Warm Up: The Last Word

Yesterday was the Presidential Debate for Math Circle! Our two candidates, Isaac and Derek, had to prove that they would make a great president. The debate started at 4:00 PM, and it ended at 4:05 PM. Isaac and Derek took turns arguing with each other. On their turn, each candidate was allowed to argue their point for 1 or 2 minutes, and then their opponent was allowed to argue for 1 or 2 minutes. This continued until the debate was over, a total of 5 minutes later. As we all know, whoever gets the last word in at the debate wins. Here is a transcript of the debate:

Debate Transcript

4:00-4:01 **Isaac**: I will make a better President, because I prefer prime numbers over the Fibonacci numbers. For example, I enjoy $2, 3, 5, 7 \ldots$

4:01-4:02 **Derek**: Lies! I tell you, the Fibonacci numbers are better than the prime numbers! I really enjoy the numbers $1, 2, 3, 5, 8, 13, 21 \ldots$

4:02-4:03 **Isaac**: May I quickly note that my opponent says that there are not more even numbers than odd numbers! But look at this list of even numbers! $2, 4, 6, 8 \ldots$

4:03-4:05 **Derek**: But they are not more numerous than the digits of π : $3.14159265359 \ldots$

So last night, Derek won the debate because he got the last word in. Of course, the night could have turned out differently if Isaac had picked a better debating strategy.

Problem 1. If Isaac and Derek hold a 10 minute debate, and each of them is only allowed to argue for one minute, and Isaac starts the debate, who will win the debate?

Since whoever finishes wins, and we have 2 participants and 10 minutes, whoever goes second wins.

So, Derek wins.

Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics .

Problem 2. Jeff and Morgan are not as good of debaters, so when they have their debate it is only 3 minutes long, and their talking points are allowed to be one or two minutes long. If Jeff starts the debate, why will he always lose? Describe Morgan's strategy.

Two possible scenarios:

Jeff starts and speaks for 1 minute or 2 minutes.

1	Jeff	1-2	Jeff
2-3	Morgan	3	Morgan

Morgan's strategy is to finish the debate by speaking for the remaining time

Problem 3. Jonathan is the moderator of the next debate between Isaac and Derek. The Debate will be 6 minutes long, and the talking points can be one or two minutes long. If he wants Isaac to win, who should he have start the debate?

since 6 is a multiple of 3, this debate will proceed in 2 sets of 3 minutes. The three minutes can play out in the in any way given above.
The person who goes second wins.
If Isaac wants to win, Derek must go first.

Problem 4. After going to debate camp, Jeff and Morgan now know how to hold longer debates. In order to show off, they decide to have a 25 minute debate. Again, they are allowed to have talking points of either one or two minutes long. If Morgan starts the debate, can Morgan win every time?

Yes.

Morgan can force the debate to behave like a 3-minute type debate by speaking for 1 minute. In the 24 minutes remaining, the person who goes second wins, which is Morgan.

	1	Morgan	1	Morgan
24 minutes remaining	2-3	Jeff	2	Jeff
	4	Morgan	3-4	Morgan
	⋮		⋮	

2. Math Games

Today we want to look at mathematical games. What are some properties that a math game should have?

- There should be a winner and a loser (no ties, and the game must stop)
- There should be no luck involved
- No secrets!

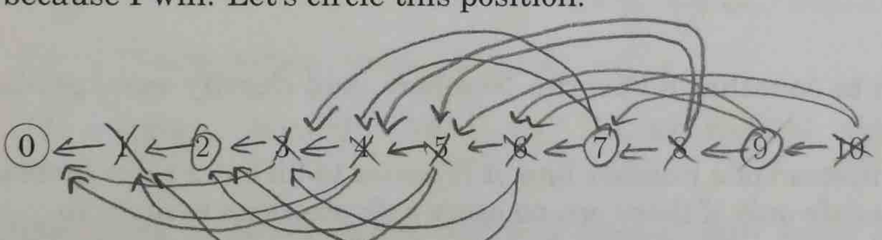
It turns out that a game is a "math game" if there is a winning strategy. A winning strategy is a method that makes you win no matter what moves your opponent may make. But how can we find the perfect strategy? Let us look at a simple game.

2.1. **Jonathan's Favorite Numbers.** In this game, we start with 10 rocks in a pile. On each turn, the player taking the turn may remove 1, 3, or 4 stones from the pile (These are Jonathan's favorite numbers). The players alternate taking turns. The player who takes the last stone wins.

(1) Let's draw a number line to represent the different number of stones that can be left in the game:

0 1 2 3 4 5 6 7 8 9 10

(2) Certainly, if I move to the position where there are 0 stones left, this is a good move, because I win! Let's circle this position.



(refer to the table on page 4 for a better idea)

(3) We draw an arrow to represent all possible moves our opponent can make after we have taken our turn and moved to a position.
Suppose we have just taken our turn and there is 1 stone left. It is now the opponent's turn. Draw the arrows corresponding to the moves that our opponent can make from position 1.

(4) Why would moving to position 1 be a losing strategy?

If I move to position 1, my opponent will pick up the 1 stone remaining and win the game. So, moving to 1 is a losing strategy for me.

(5) We put an X over a position to remember that it's a losing strategy to move to it and a circle around a position to remember that it's a winning strategy to move to it.

(6) We will fill out the rest of the number line together as a class.

2.2. **Safe and Unsafe Positions.** Let us look at the possible positions in a game: in this case, the position of the game is given by how many stones remain. We will call a position a **Safe** position (or S position) if moving to that position gives us a winning strategy. We will call a position an **Unsafe** position (or U position) if moving to that position gives the opponent a winning strategy.

Problem 5. Explain why the following properties of S and U positions are true.

- From a S position, you cannot move into another S position.

We are alternating turns. If it were possible to go from S to S, our opponent will move to S and win the game. This would mean our position was not safe.
For eg. if 5 were safe, our opponent would move to 2 (also S) and win. This means 5 is unsafe for us. This is a contradiction.

- From a U position, you must be able to move into a S position.

Similarly, if we are at an unsafe position, our opponent must be able to move to a S position and win the game. This is what makes our position unsafe.

Let's return to Jonathan's Favorite Numbers, and classify some positions as safe or unsafe.

Sometimes instead of a number line, it is easier to fill out a table instead. Remember, a position is safe only if there are no other safe positions to move to.

Position	Positions our opponent can move to	Is this position safe?
0	None	S
1	0	U
2	1	S
3	0, 2	U
4	0, 1, 3	U
5	1, 2, 4	U
6	2, 3, 5	U
7	3, 4, 6	S
8	4, 5, 7	U
9	5, 6, 8	S
10	6, 7, 9	U

Problem 6. [The Debate Game] In the debate, each player had to talk for 1 or 2 minutes. The debate lasted 10 minutes in total. Mark the moves that are possible on this number line with arrows.

0 ← 1 ← 2 ← 3 ← 4 ← 5 ← 6 ← 7 ← 8 ← 9 ← 10

Can you fill in the rest of this table to find a good strategy for the debates? Remember you win the debate if you get the last word in. I've filled in the first 4 rows.

Position	Positions our opponent can move to	Is this position safe?
0	None	S
1	0	U
2	0, 1	U
3	1, 2	S
4	2, 3	U
5	3, 4	U
6	4, 5	S
7	5, 6	U
8	6, 7	U
9	7, 8	S *
10	8, 9	U

If Jonathan starts the debate, how can he always win?

Jonathan can win by speaking for 1 minute and moving to the safe position starred above.

Problem 7. [The Long Debate] In this debate, each player is allowed to take 1 to 4 minutes off of the clock on their turn. Can you fill in the rest of this table to find a good strategy for the debates? Remember you win the debate if you get the last word in. I've filled in the first 4 rows

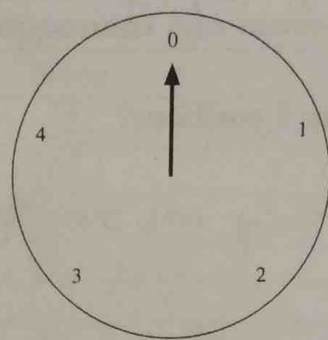
Position	Positions our opponent can move to	Is this position safe?
0	None	S
1	0	U
2	0, 1	U
3	0, 1, 2	U
4	0, 1, 2, 3	U
5	1, 2, 3, 4	S
6	2, 3, 4, 5	U
7	3, 4, 5, 6	U
8	4, 5, 6, 7	U
9	5, 6, 7, 8	U
10	6, 7, 8, 9	S
11	7, 8, 9, 10	U
12	8, 9, 10, 11	U
13	9, 10, 11, 12	U
14	10, 11, 12, 13	U
15	11, 12, 13, 14	S

If Jeff starts the debate off, is there a strategy that ensures that he wins?

NO.

Each person can only speak for 1-4 minutes, and Jeff cannot move to a S position by Speaking for 1-4 minutes.

Problem 8. [Morgan's Watch] Morgan owns a watch that he wears to the debates. When the debate starts, the watch points to 0. The debate runs for 15 minutes long, and each contestant must argue for 1 to 4 minutes on their turn.



(1) At the end of the debate, where will the clock be pointing?

0

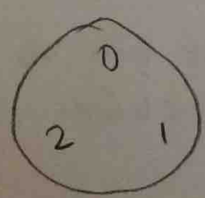
(2) If Morgan's opponent makes the first move and debates for X minutes, how long should Morgan debate for to bring the minute hand on his watch back to 0?

$5-X$

(3) What is special about the 0 position on Morgan's watch?

It's a safe position.

(4) If Morgan wants to do well at the debate where people are allowed to talk for just 1 or 2 minutes, what kind of watch should he wear? Draw it below!



Problem 9. [The Ultimate Debating Strategy] Let us look a little more at Problem 8.

(1) At what times were all of the S positions at?

0, 5, 10, 15 minutes (check the table on page 6)

(2) Was there a pattern to the S positions?

Yes, they are multiples of 5.

(3) How would you describe the S positions using mod 5 arithmetic?

$S \pmod 5 \equiv 0$

(4) Show that you cannot move from one S position to another, using mod 5 arithmetic (hint: you are subtracting off 1 to 4 minutes with every turn. What does this mean about the S positions?)

$S-1 \pmod 5 = 4$

$S-2 \pmod 5 = 3$

$S-3 \pmod 5 = 2$

$S-4 \pmod 5 = 1$

None of these positions are safe.

(5) If the debate is going to be 132 minutes long, should Morgan start off the debate if he wants to win?

Yes, Morgan should start the debate and speak for 2 minutes.

His opponent will be at position 130, which is unsafe for him.

3. The Principle of Symmetry

The game of Tabletiles is played on a circular table. On each turn, a player places a penny on the table. The players alternate turns. They are not allowed to have two pennies touch in any way. The player who places the last penny wins. I claim that the first player always wins! We will show how this is done.

Strategy: To beat this game we use the symmetry strategy to show that the first player can always win. On the first turn, the first player places a penny in the exact center of the table. Then the second player can place their penny anywhere they would like. The first player keeps on copying the second players move by placing their penny the same distance away from the central penny as the previous penny, but on the opposite side of the central penny.

Why is this a winning strategy?

There will always be an even no. of pennies around the central penny because 2 people are placing pennies one by one. So, whoever puts the second (or last) penny down around the central penny wins. This is the same who put the first (central) penny down.

3.1. **Kayles.** In the game of Kayles, a row of bowling pins are lined up in a row On your turn, you may throw the bowling ball at the pins, knocking down one or two pins that are right next to each other. A player wins when they bowl down the last pin.

1 2 3 4 5
O O O O O

For instance, the first player might knock down the second and third pin.

O O O

Then the second player might knock down the right two pins

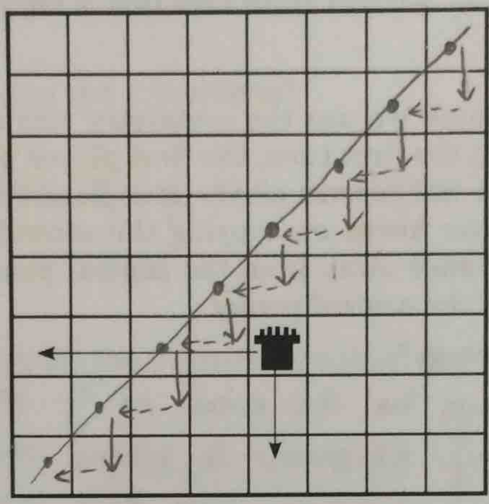
O

Then the first player could knock down the first pin and win.
How can we use the symmetry strategy to beat this game?

The first player should knock down pins 3, or 2,3, or 3,4.

4. Wyt's Rooks and Queens

In the game of Wyt Rooks, players start with a rook in the top right corner of a chessboard. On each turn, a player may move the rook a single direction of their choice as long as it moves the rook closer to the bottom left corner.



Players alternate turns. The player who moves the rook into the corner wins. Why don't we turn this into a game that we can solve with numbers?

Strategy: We want to show that two games are the same. We can think of each square on the chessboard as a pair of numbers, (x, y) where x describes the horizontal position of the square and y describes the vertical distance of the square. For example, in this game the rook is at $(5, 3)$ Let us try to translate our game into this new language. The rook starts at $(8, 8)$, and will end at $(0, 0)$. On a player's turn, they can decrease the x value by moving the rook left, or they can decrease the y value by moving the rook down.

So what are the S positions? On the diagonal.

Suppose you have just made a move and the rook is on the diagonal where $x = y$. On your opponent's turn after yours, can they move to a different position along the diagonal? No.

Use the above to show that the S positions are those that are on the diagonal of the chessboard.

Say you are on the diagonal. The opponent can move the rook left, right, up or down. You can then bring the rook back to the diagonal making the opposite moves finally reaching the bottom left corner and winning.

5. Challenge: Jeff's Favorite numbers

Jeff is fascinated with the powers of two. Whenever he hosts a debates with anybody, he insists that people only debate for a power of two (2^n) number of minutes.

Position	Positions our opponent can move to	Is this position safe?
0	None	S
1	0	U
2	0, 1	U
3	1, 2	S
4	0, 3, 2	U
5	1, 3, 4	U
6	2, 4, 5	S
7	3, 5, 6	U
8	0, 4, 6, 7	U
9	1, 5, 7, 8	S
10	2, 6, 8, 9	U
11	3, 7, 9, 10	U
12	4, 8, 10, 11	S
13	5, 9, 11, 12	U
14	6, 10, 12, 13	U
15	7, 11, 13, 14	S

Can you find a pattern? Use modular arithmetic to prove your pattern works.

Pattern: $s \pmod 3 \equiv 0$

Prove:

$[s - 2^n] \pmod 3 \not\equiv 0$

$s \pmod 3 - 2^n \pmod 3$

$= 0 + (1) \pmod 3$

$= 1 \not\equiv 0$