Proof by Contradiction

LA Math Circle (Advanced)

April 3, 2016

Today we will use a proof technique called proof by contradiction. The idea is this: we can prove a statement by proving that its negation is false.

Example Suppose the vertices of a triangle are all colored either red or blue. Prove that one of the colors must be used at least twice.

Proof. Assume otherwise. That is, assume that both red and blue are used no more than once. Then there is at most one red vertex and at most one blue vertex, which makes at most two colored vertices. But all three vertices were assigned a color, so we have a contradiction, and our assumption must have been false.

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Say you really need to get into a building, and you're looking at it, but you don't see any way to get in. What do you do? Naturally, you walk around the building to see if there is a door on one of the other sides. In the same way, proof by contradiction can allow us to look at a problem from a new angle and give us a good starting point for solving it.

Problem 1 The product of two positive numbers is greater than 75. Prove that at least one of them is greater than 8.

Problem 2 The product of a two-digit number with 5 is a two-digit number. Prove that the tens digit of the original number is 1.

Recall that a rational number is a number that can be represented as the quotient of two integers.

Problem 3 Prove that if a, b are rational numbers, then so are ab, $a+b$, and $a-b$.

Problem 4 Prove that if a is rational and b is irrational (= not rational), then $a+b$ is irrational.

Problem 5 Explain why proof by contradiction might be well-suited to proving that a number is irrational.

Problem 6 Prove that the square root of two is irrational (Hint: use the fact that a fraction can be written in reduced form so that the numerator and denominator greatest common divisor 1).

I told you last quarter to never, ever forget the following property of prime numbers: If p is a prime and $p|ab$ where a, b are integers, then either $p|a$ or $p|b$. Use this to solve the next problem.

Problem 7 Show that the square root of a prime number is irrational.

Problem 8 Prove that if a, b, c are integers and they are the side lengths of a right triangle, then abc is even.

Problem 9 Prove that there are no positive integers a, b so that $b^2+b+1 = a^2$ (Hint: $a^2 - b^2 = (a - b)(a + b)$).

Time for a break from number theory? Let's do some geometry!

Problem 10 Prove that if a, b, c are the side lengths of a triangle, then $a + b > c$.

Problem 11 Given a line ℓ and a point P not on ℓ, show that there exists a point Q on ℓ such that $PQ \perp \ell$ (that is, show how to find such a Q with a ruler and compass).

Problem 12 Use problem 11 and proof by contradiction to show that given a circle and a line tangent to the circle, the radius of the circle which passes through the point of tangency is perpendicular to the tangent line.

Ok, let's finish off with some more number theory.

Problem 13 Prove by induction that any integer $n > 1$ can be expressed as a product of prime numbers.

Problem 14 Let $p_1, p_2, \ldots, p_n$ be prime numbers. Show that the number $p_1 p_2 \cdots p_n + 1$ is not divisible by any of the numbers $p_1, p_2, \ldots p_n$.

Problem 15 Use problems 13 and 14 to prove that there are infinitely many prime numbers.

Problem 16 Prove that in problem 13, the expression of n as a product of prime numbers is unique.

Problem 17 Use problems 13 and 16 to prove that if a positive integer is not a perfect square, then its square root is irrational.