Warm-up

1. Two players take turns putting pennies on a round table so that they do not touch one another. The player who cannot place a penny loses. Does the first or second player have a winning strategy? Can you find one?

2. Everyone in the class will write down an integer between 0 and 100, inclusive, on a slip of paper. Any answer closest to (2/3) of the average of all the numbers written down is a winner. Think carefully about what you submit!

Subtraction Game / Nim

3. Recall in the subtraction game we discussed, each player must remove 1, 2, 3, or 4 coins from the pile. This makes the winning strategy to remove enough coins to leave exactly a multiple of 5 if possible.

Suppose we start with two piles instead of just one and use (n_1, n_2) to indicate the piles have n_1 and n_2 coins remaining. Now each player must remove 1-4 coins from a single pile, and the winner is the one to reach state $(0, 0)$. Determine if the first or second player has a winning strategy given the following initial state and describe it:

(a) (5, 5)
(b) (5, 7)
(c) (2, 2)
(d) (3, 4)

4. Let a and b be positive integers with binary represenation $(a_m \dots a_0)_2$ and $(b_m \dots b_0)_2$ respectively. Define the Nim-sum of a and b as the integer with binary representation $(c_m \dots c_0)_2$ where

$$c_k = \text{remainder of } a_k + b_k \text{ when divided by 2.}$$

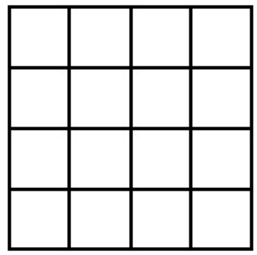
Denote this as $c = a \oplus b$.

Recall in the game Nim, players can remove any number of coins from a single pile. Parts (c) and (d) of question 3 are equivalent to playing Nim with that initial state. As discussed, the winning strategy for the first player (if there is one) is to move to states with Nim-sum 0. Does this agree with your results in parts (c) and (d)? For practice, find a winning move from the state $(7, 11, 13)$.

Chomp

5. In Chomp, players take turns biting off a chunk of a rectangular bar of chocolate that is divided into squares. A legal move consists of picking a square and removing all squares above and to the right of the original square. Here the losing player is the one who removes the bottom left square.

(a) In any square $n \times n$ initial state, find a winning strategy for the first player.



(b) Consider a $2 \times \infty$ initial state, where we interpret this as extending infinitely only to the right. Which player has a winning strategy? Describe it.

(c) Now consider the case $\infty \times \infty$ (extending infinitely up and to the right). Answer the same questions.

6. In Chomp, the first player has a winning strategy from any (finite) initial rectangle larger than 1×1. Prove this claim. (We do not need to explicitly describe a winning strategy for any initial state, but the claim can still be proved.)

Challenge Problems

1. What happens if we play the subtraction game where the number of coins removed must be a power of 2 $(0, 1, 2, 4, \ldots)$? Find the initial number of coins for which the first and second player have winning strategies.

2. Consider a row of coins initially randomly distributed between heads and tails. A valid move consists of flipping either one coin from heads to tails or flipping two adjacent coins, the right-most one which was heads. The player to make the last valid move wins. Analyze some simple starting configurations and try to generalize.

3. Consider Nim on the first quadrant in two dimensions. Here some coins sit on some initial lattice positions and can be moved anywhere directly left or directly down. Whoever moves the final coin to $(0, 0)$ wins. Play with two coins, four coins, etc. and try to come up with some strategies.