Graphs I

Advanced Math Circle
Based on a handout by Olga Radko

February 26, 2016

1. Suppose that the values of n are such that the following graphs all exist. If you get stuck, try drawing out some examples until you find a pattern.

   (a) Suppose that we had a village with n houses, and every house in the village is connected to one other house by a road. How many roads must there be total?

   (b) Same question as a before, but this time every house is connected to at least 4 different houses.

   (c) Finally, how many roads must you have if every house is connected to m other houses, where $m < n$?

   (d) You are working for the utility company in Smallville USA. In total,there are 15 telephones in Smallville, and you would like to connect the phones, so that every phone is connected to exactly 5 other phones. Is this possible? If so, draw a supporting picture. If not, explain why not.

2. Now we are going to try and prove some theorems about graphs.

   (a) I am thinking of a graph, where the first vertex has degree 5, the second one has degree 4, the third and fourth edge both have degree 3, the fifth and sixth vertex have degree 2, and the last vertex has degree 1. How many edges dopes my graph have total?

   (b) Can you relate the total number of edges to the sum of the degrees of all of the edges? In other words, if I know that vertex 1 has degree d_1, vertex 2 had degree $d_2, \ldots$ and vertex n had degree d_n, how could I relate $\sum_{i=1}^{n} d_i$ to the number of edges in the graph?

   (c) Is the sum of the degrees of the vertices of a graph even or odd?

   (d) A vertex is called **even** if is has an even degree (so has degree 0, 2, 4, etc.). A vertex is called **odd** if it has an odd degree. Prove that the number of odd vertices in a graph must be even.

3. Now, let's apply the above theorem and try and find out of some of the following graphs can exist. In the following example, if person a is friends with person b, then b is also friends with person a.

   (a) There are 30 students total in a class. Can it happen that 9 of them have 3 friends each, eleven have 4 friends each, and the remaining 10 have 5 friends each?

   (b) There are 30 students in a class. Is it possible that 20 of them have 19 friends each, and 10 of them have 9 friends each?

   (c) Prove that the number of people who have ever lived on earth, and who have shaken hands with an odd number of people in their lives, is even.

   (d) Can you have 9 line segments in the plane, such that each of them intersect with exactly 3 other? If it's possible, draw a picture proving that it is. If it isn't, prove that it isn't!

   (e) Take a 4x4 chess board, and delete the corner squares. Can you put a knight on one of the squares, and have it jump around such that it visits every square exactly once, and then returns to the square where it started?

4. For the following questions, I am going to give you a list of whole, non-negative numbers, and you are going to tell me if it's possible to have a graph where the first vertex has degree of the first number, the second vertex has the degree of the second number, etc... If such a graph exists, draw an example. If it isn't, explain why it can't exist
As an example, if the list was: 3 2 2 1, then I would be asking if there is a graph where one vertex had degree 3, two vertices had degree 2, and the last vertex has degree 1.

   (a) 4, 4, 4, 4, 4

   (b) 4, 3, 2, 2, 1

   (c) 8, 3, 3, 2, 2, 2

   (d) 5, 5, 2, 2, 2, 2

   (e) 1, 1, 1, 1, 1, 1, 1, 1, 1

   (f) 6, 6, 6, 2, 2, 2, 2

5. A graph is called complete, if every vertex is connected to every other vertex. A complete graph with n vertices is often abbreviated as K_n

   (a) Draw K_1, K_2, K_3, K_4, and K_5.

   (b) Notice, that you can draw K_3 such that none of the edges cross. Can you draw K_4 such that none of the edges cross? What about K_5?

   (c) Start with the graph of K_4, and color each edge either red or blue. Try to color the edges such that you don't have any three edges which form a triangle, and are all of the same color. Can you do it?

   (d) Same question as above, but this time try to find a similar edge coloring for K_5.

   (e) Prove that you can't find such an edge coloring for K_6.