Complex Numbers and Geometry

LA Math Circle
High School II
Dillon Zhi

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Warm-up exercises

1. Expressing a complex number z in Cartesian coordinates as $x + iy$ (where x and y are real numbers), we call x the real part of z and y the imaginary part of z, and write $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

   a. Show that $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, geometrically and algebraically.

   b. Find a similar formula for $\operatorname{Im} z$ in terms of z and $\bar{z}$.

   c. Recall that $e^{it} = \cos t + i \sin t$. Write down formulas for $\cos t$ and $\sin t$ in terms of e^{it} and e^{-it}.

2. Explain why

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

geometrically and algebraically.

3. Let $a \in \mathbb{C}$ and $\theta \in \mathbb{R}$. Derive a formula for the function which takes any point z and rotates it by the angle θ about the point a.

4. a. Suppose that $z_1 + z_2 + \cdots + z_n = 0$. Give a geometric interpretation of this equation.

   b. For any positive integer n, the numbers $1, e^{2\pi i/n}, e^{2\pi i \cdot 2/n}, \ldots, e^{2\pi i \cdot (n-1)/n}$ are called the n-th roots of unity. (Why?) Show geometrically that the sum of the n-th roots of unity is 0.

5. We can view trigonometric identities as coming from the rule for multiplying complex numbers. For example:

$$
\begin{aligned}
&\cos(\alpha+\beta) + i\sin(\alpha+\beta) \\
&= e^{i(\alpha+\beta)} \\
&= e^{i\alpha}e^{i\beta} \\
&= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta).
\end{aligned}
$$

Thus

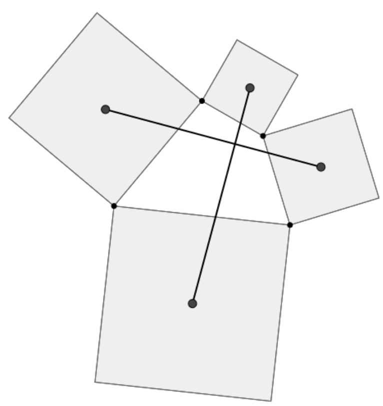
$$
\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta.
\end{aligned}
$$

Using a similar method, express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

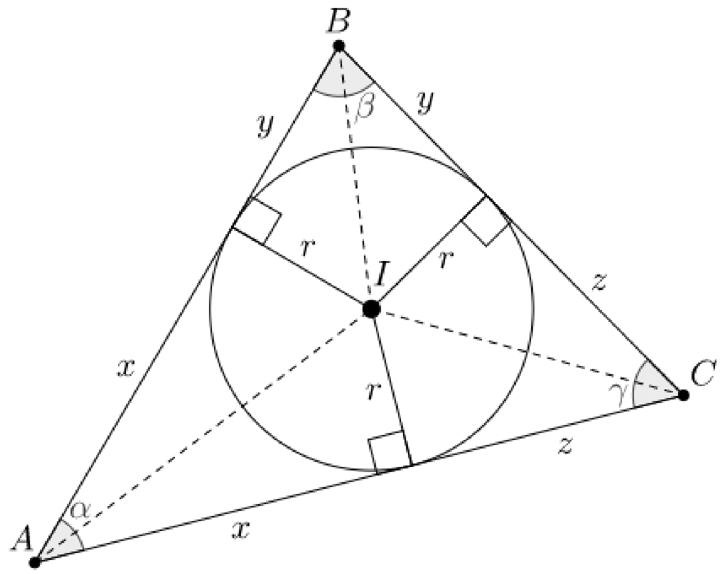
Problems

6. Show that the midpoints of any quadrilateral form a parallelogram.

7. Place squares on the edges of a quadrilateral, as shown, and connect the centers of the opposite squares by line segments. Show that these two line segments are perpendicular and have equal length.



8. In the following series of exercises we will give a derivation of Heron's formula.



a. In the figure, circle I is inscribed in triangle ABC. Find the area of triangle ABI in terms of x, y, and r. Similarly, find the area of triangle BCI and the area of triangle CAI. Use this to show that the area of triangle ABC is rs, where $s = x + y + z$ is the "semiperimeter" of the triangle.

b. Find $\arg(x+ir)$. Do the same for $y+ir$ and $z+ir$. Use this to show that $(x+ir)(y+ir)(z+ir)$ is purely imaginary (that is, its real part is 0).

c. Compute the real part of $(x+ir)(y+ir)(z+ir)$. Now show that $sxyz = (rs)^2$.

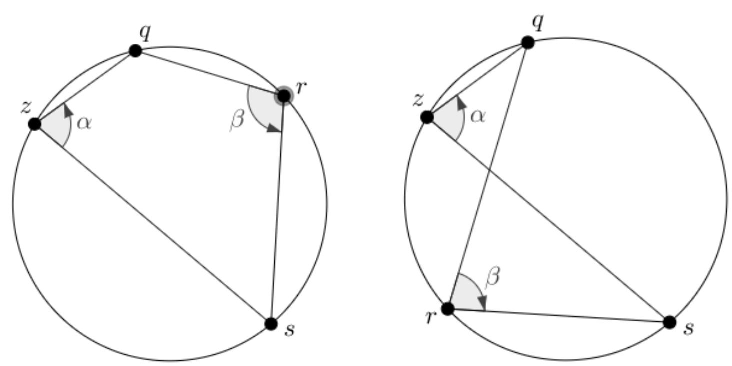
d. Let $a = BC$, $b = CA$, and $c = AB$. Express s, x, y, and z in terms of a, b, and c. Put all this together to get Heron's formula, which expresses the area of a triangle in terms of the lengths of its sides:

$$\text{Area} = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

9. Define the "cross-ratio" of four complex numbers z, q, r, and s to be

$$[z, q, r, s] = \frac{\frac{z-q}{z-s}}{\frac{r-q}{r-s}}.$$

a. Show that $\arg[z, q, r, s] = \alpha + \beta$ in the figures. In particular, the points z, q, r, and s are concyclic (lie on a common circle or line) if and only if $\alpha + \beta$ is 0 or 180°, so $[z, q, r, s]$ is real if and only if z, q, r, and s are concyclic.



b. Show that $[\frac{1}{z}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}] = [z, q, r, s]$. In particular, use this to show that the image of every circle or line under the mapping $f(z) = 1/z$ is a circle or line.

10. Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then R_2, $\ldots$, then R_n. For any arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

11. a. Let a, b, and c be points on the unit circle. (That is, they form a triangle inscribed in the unit circle.) Show that the orthocenter of the triangle is the point $a + b + c$.

b. Let $ABCD$ be a cyclic quadrilateral. Let H_A, H_B, H_C, and H_D denote the orthocenters of triangles BCD, CDA, DAB, and ABC, respectively. Show that the lines AH_A, BH_B, CH_C, and DH_D are concurrent.

12. a. A particle moves in the plane, with position $z = \cos(t - ic)$ at time t. (Here, c is some positive real constant.) Show that the particle's trajectory traces out an ellipse, with the origin at its center.

b. Another particle moves in the plane, with position $w = z^2$. Show that the particle's trajectory traces out an ellipse, with the origin as a *focus*.