

Parity, and Other Problems

1. A Donkey, a Dog, a Cat, and a Rooster form a tower by standing on each other's shoulders. One possibility (probably the most comfortable) is Donkey on bottom, then Dog, Cat, and Rooster. How many different towers are possible?

2. A combination lock has a circle with notches labeled from 0 to 39. A valid combination is three numbers; the first number must be even; the second and third numbers can be anything, except the second number has to be at least two notches away from the first number, and the third must be at least two notches away from the second number. How many valid combinations are there?



3. A domino will cover two adjacent squares of a chessboard, horizontally or vertically. If we remove the squares on two opposite corners of the chessboard, is it possible to cover the remaining squares with no overlap?

4. Twenty-one ninjas sit in a circle. They will choose their leader by going around the circle, and having every second ninja leave the circle until only one remains. So ninja #1 stays, then ninja #2 leaves the circle, ninja #3 stays, ninja #4 leaves. It goes on this way around the circle until ninja #21, who stays. The next ninja is ninja #1 again, who now leaves. The next ninja still in the circle is ninja #3 (since ninja #2 already left), so ninja #3 stays. Then ninja #5 leaves, ninja #7 stays, and so on. Who will be the last ninja standing?

5. You're in charge of arranging the pedestals for the 1st, 2nd, and 3rd place winners of the Pullington County Tug-of-War contest. The pedestals are currently arranged in the order 1-2-3, but you want to make them 2-1-3. Since they're quite heavy, you've asked Rosie the Robot to help move them. Unfortunately, Rosie's malfunctioning and will only respond to commands which tell it to switch the position of two pedestals, and then switch another two. Which arrangements are possible using Rosie to make as many of this type of move as you like?



6. In chess, a knight moves in an L shape, going two squares in any direction and then one square in a perpendicular direction.
- On a chessboard, a knight starts from square $a1$ and travels to square $a2$. What is the fewest number of moves required? What is the fewest number of moves from $a1$ to $h8$?
 - A knight starts on square $a1$ and returns there after making several moves. Show that the knight makes an even number of moves.
 - Can a knight start at square $a1$ and go to square $h8$ visiting each of the remaining squares exactly once on the way?
7. 100 checkers are placed in a row. On each move one can exchange any pair of checkers that has exactly one checker between them. (For example, one can exchange the 1st and 3rd checkers, or 8th and 10th, etc.) Can one reverse the order of checkers by performing this operation many times?
8. A cube has all vertices and centers of faces marked. The diagonals of each of the sides are also drawn. Can one go along the (parts of) diagonals and visit each of the marked points exactly once?
9. A king is placed somewhere on a 4×4 chessboard. Two players take turns moving the king around the chessboard. According to chess rules, a king can only move to an adjacent square (up, down, left, right, or diagonal). In addition, in this game the king is not allowed to return back to the square it just came from. The winner of the game is the first person who moves the king to any square it has already visited. Which player (the starting player or his opponent) can always guarantee a win in this game? Describe the winning strategy.

Now what if the chessboard is the usual 8×8 ? Identify the winning player and describe their strategy.