CRACKING THE 15 PUZZLE - PART 4: TYING EVERYTHING TOGETHER

BEGINNERS 02/21/2016

Review

Recall from last time that we proved the following theorem:

Theorem 1. *The sign of any transposition is* -1.

Using this theorem, we were able to solve the last few questions in the handout. We will go over these as a class:

Problem 1. Suppose we have a 15 puzzle where the distance between the blank square and the lower-right corner is an odd number. Is the number of transpositions we need to apply to the puzzle to move the blank square to the lower-right corner odd or even?

Each "move" either increases or decreases the taxicab distance by one, so the first move
↳ which corresponds to a transposition
would make the distance even, the second odd, etc.
Since we will eventually decrease the distance to zero in our solution, which is even, we must make an odd number of moves / transpositions

Problem 2. Suppose we have a 15 puzzle with an odd number of inversions. How many transpositions do we need to apply to the puzzle so that there are no inversions?

Each move/transposition changes the sign of the puzzle, so the first move/transposition would change the number of inversions to even, the second move to odd, etc.
Since we will eventually want no inversions in our solution, which is even (as zero is even), we will need to make an odd number of moves.

Problem 3. Do you think a solution for Loyd's puzzle, as shown below exists? Explain your answer.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

This puzzle has an odd number of inversions, but an even taxicab distance from the blank space to the lower-right corner. The number of inversions tells us there has to be an odd number of moves made to get to the solution, and the taxicab distance tells us there has to be an even number of moves made to get to the solution. We can't perform both an even and odd number of moves, so no solution is possible.

Based off of our answer in Problem 3, we can now construct and prove a second theorem:

Theorem 2. *Configurations of the 15 puzzle have "similar parity" when the number of inversions and the taxicab distance from the blank space to the lower right corner are both odd or both even. Configurations of the 15 puzzle have "opposite parity" otherwise. Configurations of the 15 puzzle with opposite parities cannot be solved.*

Proof. Configurations of the 15 puzzle with opposite parity cannot be solved.

Configurations of the 15 puzzle with opposite parity have different parities for the number of inversions and the taxicab distance. Without loss of generality (in other words, we will still reach the same conclusion if the parities we assume in the next step were switched), assume that the number of inversions in the configuration was odd. Then the taxicab distance from the blank space to the lower-right corner would have to be even because the configuration has opposite parity.

Based on Theorem 1, the number of inversions tells us that an odd number of moves must be made to obtain the solution.

Furthermore, the taxicab distance tells us that an even number of moves must be made to obtain the solution.

The opposite parities imply that both an odd and even number of moves must be made to obtain a solution. This is impossible to do, so obtaining a solution is therefore impossible.

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Are We Done? (Some Questions About Logic)

Problem 4. We know from Theorem 2 that 15 puzzles with configurations of opposite parity cannot be solved. Do you think we can now determine whether or not there's a solution for all configurations of the 15 puzzle? Why or why not?

No. We have not shown anything about whether or not configurations with similar parity can be solved, so we would not know whether or not a solution exists for a configuration of the 15 puzzle with similar parity.

Regardless of your answer, we need some logic to figure out what the right answer to Problem 4 is.

We use the symbol $\implies$ to mean "implies that".

If we had two statements, P and Q, $P \implies Q$ means that IF P is true, THEN Q is true.

Example. Theorem 2 can be thought of as an if-then statement, where P and Q are the following:

P : "The configuration of the 15 puzzle has opposite parity."

Q : "The configuration of the 15 puzzle cannot be solved."

This gives the following statement:

"The configuration of the 15 puzzle has opposite parity $\implies$ the configuration of the 15 puzzle ~~can~~ cannot be solved."

Which is equivalent to

"If the configuration of the 15 puzzle has opposite parity, then the configuartion of the 15 puzzle ~~can~~ cannot be solved."

Example. Some more if-then statements are shown below:

- It is 100 degrees outside $\implies$ It is hot outside
- Goldie is a dog $\implies$ Goldie is an animal
- It is raining $\implies$ There is traffic

Problem 5. Come up with your own if-then statement below.

No one went to the carnival $\implies$ Anna did not go to the carnival

The *contrapositive* of an if-then statement is when we negate each statement and turn the if-then statement around. We use the ! symbol to mean "not".

So the contrapositive of $P \implies Q$ is $!Q \implies !P$.

Problem 6. The contrapositive to the statements given in the above examples are shown below. If the statements in the examples above are true, are each of the contrapositives shown below true?

- It is not hot outside $\implies$ It is not 100 degrees outside

  True. If 100 degrees means it's hot, then being not hot means it can't be 100 degrees. (otherwise it would be hot)

- Goldie is not an animal $\implies$ Goldie is not a dog

  True. If Goldie WERE a dog, then Goldie would be an animal. However, Goldie is not an animal, so Goldie can't be a dog.

- There is no traffic $\implies$ It is not raining

  True. If it WERE raining, then there would have to be traffic. However, we said there isn't traffic, so it can't be raining.

Logically, all statements are equivalent to their contrapositive.

Problem 7. What is the contrapositive to Theorem 2? (Check your answer with your assistant instructor to make sure it's correct.)

The configuration of the 15 puzzle is solvable $\Rightarrow$ the configuration doesn't have opposite parity.

The *converse* of an if-then statement is when we turn the if-then statement around without negating each statement.

So the converse of $P \implies Q$ is $Q \implies P$.

The converse of the statement is NOT always true.

Problem 8. The converse to the statements given in the above examples are shown below. If the statements in the examples above are true, are each of the converses shown below true? Give a counterexample to show how the statements below could be wrong.

- It is hot outside $\implies$ It is 100 degrees outside

  No. It could be 101 degrees, and still be hot outside.

- Goldie is an animal $\implies$ Goldie is a dog

  No. Goldie could be a goldfish.

- There is traffic $\implies$ It is raining

  No, there could be traffic for other reasons as well, such as there being a traffic accident.

Problem 9. What is the converse to Theorem 2? (Check your answer with your assistant instructor to make sure it's correct.)

The configuration of the 15 puzzle is unsolvable $\implies$ the configuration of the 15 puzzle has opposite parity

Because not all converses are true, we have to prove that the converse is true in order to use it.

Problem 10. If we knew that the configuration of the 15 puzzle had similar parity, could we conclude that the 15 puzzle was solvable? Explain your answer.

No. The statement shown is the contrapositive to "if the configuration is unsolvable, then the
so they're logically equivalent
configuration must have opposite parity", which is the converse of Theorem 2. However, we have just shown that converses do not follow logically from the statements, so we would have to prove that it's true before assuming it.

Theorem 3. *Any ~~even~~ configuration of the 15 puzzle with similar parity is solvable.*

Because Theorem 3 is the converse of Theorem 2, we have to solve it in order to use it. Theorem 3 is not hard to prove using mathematical induction. However, we are not going to do it at the moment as induction proofs are a little bit beyond what we have learned. For now, we can take it as a fact without proving it.