Infinity II

It's as easy as $\aleph_1, \aleph_2, \aleph_3$

Math Circle

February 5, 2016

Let's get right into it! The very last thing that we did last week, was we showed that there was a bijection between the natural numbers $\mathbb{N}$ and $\mathbb{Z}^2$ (Remember, $\mathbb{Z}^2$ is the set of all pairs of numbers (a, b), where a and b are integers). You need to use this bijection to answer the following two questions.

1. There is an enemy among us. As we speak, there is a submarine somewhere on the number line. We don't know where it started, only that it started at some integer, and that is is moving with a constant integer velocity. For example, it could have started at $+4$ and could have a velocity of 1, or it could have started at -2502351 and it could be moving with a velocity of -2359. Every second we can release a depth charge at an integer, and it will immediately explode and destroy the sub if it is there. Can you come up with a plan to eventually destroy the submarine?

2. Can you find a bijection between the rational numbers (the fractions) and the natural numbers?

So at this point, we have shown that a whole bunch of sets are in bijection with the natural numbers. At this point, you might be thinking, can we put every infinite set in bijection with $\mathbb{N}$? If two sets are infinite, then they have to be the same size, right...? We'll answer that question today, but first, let's find some bijections between parts of the real number line, $\mathbb{R}$, which is all of the numbers, positive, negative, rational, irrational, everything!

1. Can you find a graphical bijection between the line segment $[0, 1]$ and $[0, 2]$? I know that you can find a formula to do it, but I want you to use a geometric argument.

2. Using a picture, show that there is a bijection between the points in $[0, 1]$ and between any finite segment of the real line, that is $[a, b]$ for any $a, b \in \mathbb{R}, a < b$

3. Using a picture, can you find a bijection between all of the points on the real like and only the points in (0,1)?

4. Do you think that there is a bijection between $\mathbb{R}$ and $\mathbb{R}^2 = (\mathbb{R}, \mathbb{R})$?

Now we are going to try and prove that $|\mathbb{N}| < |\mathbb{R}|$. In order to prove this, we have to prove that there aren't any bijections between $\mathbb{N}$ and $\mathbb{R}$. How could we possibly prove something like that? Well, one way is to use Cantor's diagonalization argument, considered by some to be the most elegant and important proof in all of math.

1. Prove that there does not exist a bijection between $\mathbb{N}$ and $[0, 1)$.

The trick is this, suppose that we had a candidate for a bijection, then we could write it like this:

$$\begin{array}{ccl} \mathbb{N} & & [0,1) \\ 1 & \rightarrow & .032591\ldots \\ 2 & \rightarrow & .510539\ldots \\ 3 & \rightarrow & .000034\ldots \\ 4 & \rightarrow & .328641\ldots \\ 5 & \rightarrow & .889813\ldots \\ 6 & \rightarrow & .141599\ldots \\ \vdots & & \vdots \end{array}$$

Now look at the $[0, 1)$ column and focus on only the number along the diagonal,

$$\begin{array}{ccl} \mathbb{N} & & [0,1) \\ 1 & \rightarrow & .\mathbf{0}32591\ldots \\ 2 & \rightarrow & .5\mathbf{1}0539\ldots \\ 3 & \rightarrow & .00\mathbf{0}034\ldots \\ 4 & \rightarrow & .328\mathbf{6}41\ldots \\ 5 & \rightarrow & .8898\mathbf{1}3\ldots \\ 6 & \rightarrow & .14159\mathbf{9}\ldots \\ \vdots & & \vdots \end{array}$$

Let m be the number in $[0, 1)$ which is formed by taking every digit on the diagonal, so for this candidate bijection, $m = .010619\ldots$ Now, let $\tilde{m}$ be the number that you get if you add 1 to each digit of m modulo 10, so $\tilde{m} = .121720\ldots$ What can you say about $\tilde{m}$? Can it appear in the right hand side of the list? What does this tell you about this candidate bijection?

Use this to prove that there is no bijection between $\mathbb{N}$ and $[0, 1)$.

The above proof shows that $|\mathbb{N}| < |\mathbb{R}|$, that is, that $\mathbb{N}$ has a smaller cardinality than $\mathbb{R}$, in layman's terms, there are more real numbers then natural numbers. This proves that just because two sets are both infinite, they are not the same cardinality. In other words, there are at least 2 sizes of infinity, the size of the $\mathbb{N}$, and the size of the $\mathbb{R}$. Are there even more infinities? If so, how many total?

1. Prove if you have an infinite set $\mathcal{X}$, then $|\mathcal{X}| < |P(\mathcal{X})|$, where $P(\mathcal{X})$ is the power set of $\mathcal{X}$, in other words it is the set of all subsets of $\mathcal{X}$, infinite sets included. As an example, if $\mathcal{X} = \mathbb{R}$, then $P(\mathbb{R})$ would contain $\{1, 266, \pi, -e, 5\}$, $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\}$, and $\mathbb{R}$ itself.

2. Do you think that there is a set $\mathcal{Y}$ such that $|\mathbb{N}| < |\mathcal{Y}| < |\mathbb{R}|$? Explain.

Challenge Questions

1. Can you find a bijection between $\mathbb{N}$ and $F_{\mathbb{N}\cup\{0\}}$, the set of all finite sequences of natural numbers including zero? A finite sequence of natural numbers is any sequence of natural numbers and zero that doesn't

go on forever, so $\{1,2,3,4,5\}$, $\{0,16,246981,1236178126581,0,0,0,124\}$, and $\{7\}$ are in $F_{\mathbb{N}\cup\{0\}}$.

2. Can you take an interval of length 1, cut it up into tiny pieces, and move those pieces around so that they covers every rational number? What if your initial interval is of length $\frac{1}{2}$? $\frac{1}{4}$? What does this tell you? Can you do the same thing for $\mathbb{R}$?

3. Can you find a bijection between $\mathbb{R}$ and $\mathbb{R}^n$ where n can be any natural number?

4. Can you find a bijection between $\mathcal{C}$ and $\mathbb{R}$, where $\mathcal{C}$ is the Cantor set? Ask an instructor to explain to you what the Cantor set is.

5. Can you prove that $|\mathbb{R}| < |\mathcal{F}|$ where $\mathcal{F}$ is the set of all real valued functions?