

CRACKING THE 15 PUZZLE - PART 3: APPLYING PERMUTATIONS AND TAXICAB GEOMETRY TO PUZZLE 15

BEGINNERS 02/07/2016

Warm Up

Write the following composition of transpositions as a single permutation on a set of five elements.

$$(1) \sigma = (5 \ 3) \circ (5 \ 2)$$

$$(1 \ 2 \ 3 \ 4 \ 5) \xrightarrow{(5 \ 2)} (1 \ 5 \ 3 \ 4 \ 2) \xrightarrow{(5 \ 3)} (1 \ 5 \ 2 \ 4 \ 3)$$

$$(2) \sigma = (5 \ 2) \circ (3 \ 2) \circ (3 \ 1)$$

$$(1 \ 2 \ 3 \ 4 \ 5) \xrightarrow{(3 \ 1)} (3 \ 2 \ 1 \ 4 \ 5) \xrightarrow{(3 \ 2)} (3 \ 1 \ 2 \ 4 \ 5) \xrightarrow{(5 \ 2)} (3 \ 5 \ 2 \ 4 \ 1)$$

$$(3) \sigma = (4 \ 3) \circ (4 \ 2) \circ (3 \ 1) \circ (5 \ 4)$$

$$(1 \ 2 \ 3 \ 4 \ 5) \xrightarrow{(5 \ 4)} (1 \ 2 \ 3 \ 5 \ 4) \xrightarrow{(3 \ 1)} (3 \ 2 \ 1 \ 5 \ 4) \xrightarrow{(4 \ 2)} (3 \ 5 \ 1 \ 2 \ 4) \xrightarrow{(4 \ 3)} (3 \ 5 \ 2 \ 1 \ 4)$$

Write the following permutations as a composition of transpositions. Are there an odd or even number of transpositions?

(1) $\sigma = (3 \ 2 \ 4 \ 1)$

$$(1 \ 2 \ 3 \ 4) \xrightarrow{(3 \ 1)} (3 \ 2 \ 1 \ 4) \xrightarrow{(4 \ 3)} (1 \ 3 \ 2 \ 4) \xrightarrow{(4 \ 1)} (1 \ 3 \ 2 \ 4 \ 1)$$

$$\sigma = (4 \ 3) \circ (3 \ 1) \quad \text{even.}$$

(2) $\sigma = (2 \ 1 \ 5 \ 3 \ 4)$

$$(1 \ 2 \ 3 \ 4 \ 5) \xrightarrow{(2 \ 1)} (2 \ 1 \ 3 \ 4 \ 5) \xrightarrow{(5 \ 3)} (2 \ 1 \ 5 \ 4 \ 3) \xrightarrow{(5 \ 4)} (2 \ 1 \ 5 \ 3 \ 4)$$

$$\sigma = (5 \ 4) \circ (5 \ 3) \circ (2 \ 1) \quad \text{odd}$$

(3) $\sigma = (3 \ 5 \ 2 \ 1 \ 4)$

$$(1 \ 2 \ 3 \ 4 \ 5) \xrightarrow{(3 \ 1)} (3 \ 2 \ 1 \ 4 \ 5) \xrightarrow{(5 \ 2)} (3 \ 5 \ 1 \ 4 \ 2) \xrightarrow{(5 \ 3)} (3 \ 5 \ 2 \ 4 \ 1) \xrightarrow{(5 \ 4)} (1 \ 3 \ 5 \ 2 \ 4)$$

$$\sigma = (5 \ 4) \circ (5 \ 3) \circ (5 \ 2) \circ (3 \ 1) \quad \text{even.}$$

For the following permutations, list the inversions in the permutations and calculate the sign of the permutations.

$$(1) \sigma = (3 \ 2 \ 4 \ 1)$$

$$(3 \ 2), (3 \ 1), (2 \ 1), (4 \ 1)$$

$$(-1)^4 = 1$$

$$(2) \sigma = (2 \ 1 \ 5 \ 3 \ 4)$$

$$(2 \ 1), (5 \ 3), (5 \ 4)$$

$$(-1)^3 = -1$$

$$(3) \sigma = (3 \ 5 \ 2 \ 1 \ 4)$$

$$(3 \ 2), (3 \ 1), (5 \ 2), (5 \ 1), (5 \ 4), (2 \ 1)$$

$$(-1)^6 = 1$$

Another Way of Calculating the Sign of a Permutation

We say that a permutation is *even* if the sign is 1 (there is an even number of inversions) and *odd* if the sign of the permutation is -1 (there is an odd number of inversions). Whether a permutation is even or odd is referred to as the *parity* of a permutation.

Problem 1. Are the following transpositions even or odd?

- (1) $\sigma = (5 \ 2)$ on a set of 5 elements.

$(1 \ 5 \ 3 \ 4 \ 2)$

$$(-1)^5 = -1 \quad \text{odd}$$

- (2) $\sigma = (4 \ 3)$ on a set of 6 elements

$(1 \ 2 \ 4 \ 3 \ 5 \ 6)$

$$(-1)^1 = -1 \quad \text{odd}$$

- (3) $\sigma = (2 \ 1)$ on a set of 3 elements.

$(2 \ 1 \ 3)$

$$(-1)^1 = -1 \quad \text{odd}$$

Problem 2. Look at the permutations given on page 2 and 3. What is the correlation between the number of transpositions of a permutation and the parity of it? Can you think of an explanation as to why this correlation exists?

Odd permutations consist of an odd number of transpositions,
even permutations consist of an even number of transpositions.

From problem 1, it seems like every transposition has
sign -1, so ~~therefore~~ maybe the sign of a permutation
can be given by the sign of its transpositions.

We will now try to prove our first theorem.

Theorem 1. *The sign of any transposition is -1 .*

Before giving Theorem 1 a formal proof, let's do a proof on a concrete example.

Suppose we have a set of 10 elements

$$(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10)$$

and we wanted to perform a transposition $(8 \ 3)$ on the elements.

Problem 3. How many elements are positioned between 3 and 8 (including 3 and 8)?

6 elements

Problem 4. What will the set look like after applying the transposition $(8 \ 3)$ on the set of 10 elements?

$$(1 \ 2 \ 8 \ 4 \ 5 \ 6 \ 7 \ 3 \ 9 \ 10)$$

Problem 5. How many inversions of the permuted set contain 8?

5 inversions

Problem 6. How many inversions in the permuted set contain 3?

5 inversions

Problem 7. How many inversions are there in total? Is this number even or odd? Remember that the inversion $(8 \ 3)$ is counted twice!

$$5 + 5 - 1 = 9$$

This is odd.

Now suppose we have a set of n elements.

$$(1 \ 2 \ 3 \ \dots \ j \ \dots \ n-2 \ n-1 \ n)$$

$$(\cancel{1} \ \cancel{2} \ \cancel{3} \ \dots \ \cancel{n-2} \ \cancel{n-1} \ \cancel{n})$$

and we wanted to perform a transposition $(j \ i)$ on the elements.

Problem 8. How many elements are positioned between i and j (including i and j)?

$$j-i+1 \quad \text{elements}$$

Problem 9. What will the set look like after applying the transposition $(j \ i)$ on the set of n elements?

$$(1 \ 2 \ 3 \ \dots \ j \ \dots \ i \ \dots \ n-2 \ n-1 \ n)$$

Problem 10. How many inversions of the permuted set contain j ?

$$j-i$$

Problem 11. How many inversions in the permuted set contain i ?

$$j-i$$

Problem 12. How many inversions are there in total? Is this number even or odd? Remember that the inversion $(j \ i)$ is counted twice!

$$(j-i) + (j-i) - 1 = 2 \times (j-i) - 1$$

This is odd because $2 \times (j-i)$ must be even

and an even number -1 gives an odd number

We will now write the formal proof together as a class:

Proof. The sign of every transposition is -1 .

Suppose we have a set of n elements and want to apply the ~~transposing~~ transposition $(j\ i)$.

The resulting permutation will look like

$$(1\ 2\ 3\ \dots\ j\ \dots\ i\ \dots\ n-2\ n-1\ n).$$

This permutation has $2 \times (i-j) - 1$ inversions, which is odd because $2 \times (i-j)$ must be even and an even number $+1$ gives an odd number.

$$\text{So } \text{sign}(j\ i) = (-1)^{2 \times (i-j) - 1} = (-1)^{\text{odd \#}} = -1.$$

~~So~~

□

Using Theorem 1, we know that each time we apply a transposition to a set, the parity of the set switches.

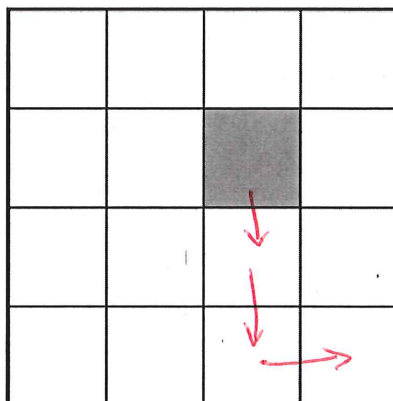
Problem 13. Without doing any calculations, what is the parity of the following permutation?

$$(4\ 3) \circ (3\ 2) \circ (3\ 1) \circ (5\ 1) \circ (4\ 3) \circ (3\ 1) \circ (6\ 1) \circ (5\ 4)$$

$$(-1)^8 = 1 \quad \text{so it's even.}$$

Applying What We Have Learned to the 15 Puzzle

Problem 14. Suppose we had a blank 15 puzzle shown in the orientation below.



- (1) What is the taxicab distance of the empty square to the lower-right corner?

3

- (2) How many moves do we have to make so that the empty square is in the lower-right corner?

3 moves

- (3) Using the terms we have learned about permutations, how many transpositions do we have to make so that the empty square is in the lower-right corner? Explain your answer.

Each move corresponds to a transposition,
so we have to perform 3 transpositions

- (4) Suppose we know that the parity of the blank 15 puzzle is ^{shown above} odd. What would be the parity of the blank 15 puzzle be once we have moved the empty square to the lower-right corner? Explain your answer.

Since we have to perform 3 transpositions,

original orientation : odd $\xrightarrow{1^{st} \text{ transposition}}$ even

$\xrightarrow{2^{nd} \text{ transposition}}$ odd $\xrightarrow{3^{rd} \text{ transposition}}$ even.

so the resulting \neq parity would be even.

- (5) Suppose we were only told that the taxicab distance between the blank square and the lower-right corner was an odd number and that the parity of the 15 puzzle was odd. What would the parity of the 15 puzzle be once we have moved the empty square to the lower-right corner?

odd + odd = even

so the parity would be even.

Problem 15. Suppose we have a 15 puzzle where the distance between the blank square and the lower-right corner is an odd number. Is the number of transpositions we need to apply to the puzzle to move the blank square to the lower-right corner odd or even?

we need to perform an odd number of transpositions to move an odd number of spaces.

Problem 16. Suppose we have a 15 puzzle with an odd number of inversions. How many transpositions do we need to apply to the puzzle so that there are no inversions?

No inversions = even number of inversions

so we have to make an odd number of transpositions to get undo the odd number of inversions.

While you might not realize it, you now have all the tools you need to determine whether Sam Loyd's problem with the 15 puzzle shown below has a solution or not. We will formally discuss this next class, but for now, we can make educated guesses.

| | | | |
|----|----|----|----|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 15 | 14 | |

Problem 17. Using your answers from Problem 15 and Problem 16, do you think a solution for Loyd's puzzle exists? Explain your answer.

The taxicab distance of the blank square to the lower right corner is even (zero) so we have to make an even number of transpositions to get the final solution. The number of inversions on the puzzle is odd, so we need an odd number of transpositions to get the final solution.

We can't make both an odd number of transpositions and an even number of transpositions at the same time, so no solution exists.