

Special Relativity - Math Circle

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Introduction: The Axioms of Special Relativity

The principle of relativity existed long before Einstein. It states:

Principle of relativity: The laws of physics take the same form in all inertial reference frames

What does it mean for the laws of physics to take the same form? It means that there is no experiment which gives different physical results when performed in different inertial frames. The word physical is important here – we'll come back to that in the next session.

The principle of relativity was initially derived as a consequence of the laws of motion inferred by Galileo and Newton in the 17th century. It wasn't until the 19th century, when physicists began to understand electricity and magnetism, that they arrived at an apparent contradiction to the principle of relativity: the fact that light always moves with a constant speed c . Most physicists thought this was incompatible with the principle of relativity, which they abandoned in favor of more convoluted solutions. Luckily for us, however, Einstein discovered a way to reconcile the principle of relativity with the constancy of the speed of light – the theory of special relativity.

Einstein took the two principles he wished to reconcile as axioms: ¹

Axioms of special relativity

1. The laws of physics take the same form in all inertial reference frames
2. The speed of light is constant in all inertial frames

From this point forward, we wish to investigate the consequences of special relativity by considering observers in different inertial frames, and how they each view certain events. This requires us to ask how coordinates in different inertial frames are related. Such coordinate transformations are called *Lorentz transformations*.

¹This, in itself, was not a novel approach. It was accepting the consequences of these two axioms as physical reality that was Einstein's true innovation.

Inertial frames are defined as moving with constant velocity with respect to one another. Combining this fact with the first axiom of special relativity, we require Lorentz transformations to be such that:

Properties of Lorentz transformations

1. Observers in two different inertial frames see each other moving with the same relative speed v
2. Observers in all inertial frames see light moving with speed c .

Let us consider an inertial frame A with coordinates x and ct , as well as another inertial frame B with coordinates x' and ct' , which moves with velocity v away from frame A . For simplicity, we will always require that the origins of the two coordinate systems coincide. That is, an event with coordinates $(x, ct) = (0, 0)$ in frame A will always have coordinates $(x', ct') = (0, 0)$ in frame B .

To make things easier, we will assert that the Lorentz transformations are linear.² That is, we have

$$x' = ax + bct \tag{1}$$

$$ct' = dx + ect \tag{2}$$

where a, b, d, e are constants which cannot depend on x, ct or x', ct' . (Note, however, that they may depend upon v and, of course, c .)

This is all the setup we need to derive the Lorentz transformations. However, rather than having you derive it now, we will give you the answer, and defer the derivation to the last problem. This is because we find it more useful for you to gain some more intuition for special relativity (by doing some problems) before you perform the derivation. This way, you should be able to better understand the physics that goes into the derivations.

The most general **Lorentz transformation** is given by

$$\begin{aligned} x' &= \gamma(x - vt) & \gamma &\equiv \frac{1}{1 - v^2/c^2} \\ ct' &= \gamma(ct - \frac{v}{c}x) \end{aligned}$$

You can gain some intuition for this result in the problems which follow.

²One can make various arguments why the Lorentz transformations are linear, but the most convincing answer to a physicist is that (1) the assumption of linearity gives a result which is consistent with the axioms, and more importantly that (2) the result agrees with experiments. Proving linearity from the axioms is much more difficult – it requires one to think very carefully about what exactly the axioms are saying, as well as exactly which types of transformations they allow.

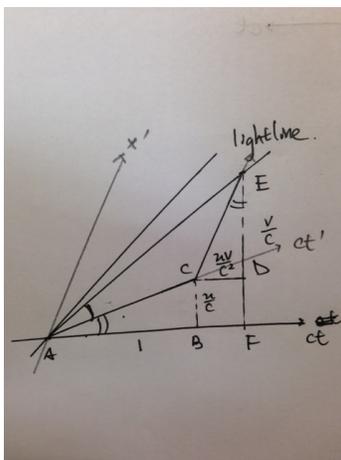


Figure 1: Vector Addition

Examples

Example 1 An important axiom of Special Relativity is that the speed of light is constant relative to any observer. However, this would mean that our usual vector addition wouldn't work, so we need to come up with a new formula. Suppose an observer O' is moving at speed u relative to observer O and observer O'' is moving at speed v relative to O' . Then what is the speed w of observer O'' relative to observer O ?

Proof. Suppose we have an observer O' moving right relative to the observer O at a speed $\beta = v/c$. Then relative to observer O , the worldline of O' can be expressed as $x = \beta ct$. However, relative to O' the worldline of itself is expressed as $x' = 0$. In that case, the worldline of O' can be seen as the ct' -axis. And because the speed of light is constant relative to any observer, the x' -axis must be the reflection of ct' about the light line.

Now I can refer to the figure as Figure 1

In the figure, AF is the ct -axis and the diagonal line is the light line. AC represents the ct' -axis and CE represents the x' -axis if an observer O' is moving relative to the observer O at a speed u . If a third observer O'' is moving relative to O' at a speed v , then AE represents its worldline. Let the length of AB be 1, then the length of BC is u/c and the length of AC is $\sqrt{1 + (u/c)^2}$. Notice that the angle $\angle ACE$ is 90 degrees relative to observer O' . Hence, the length of CE is

$$\begin{aligned} CE &= AC \cdot \tan \angle CAE \\ &= \frac{v}{c} \cdot \sqrt{1 + \left(\frac{u}{c}\right)^2} \end{aligned}$$

Notice that the triangles $\triangle ABC$ and $\triangle CDE$ are similar. Therefore, the length of ED is

$$\begin{aligned} ED &= CE \cdot \cos \angle CED \\ &= \frac{CE}{\sqrt{1 + \tan^2 \angle CED}} \\ &= \frac{v}{c} \end{aligned}$$

Similarly, the length of CD is

$$\begin{aligned} CD &= \frac{v}{c} \cdot \tan \angle CED \\ &= \frac{uv}{c^2} \end{aligned}$$

Therefore, the length of AF is

$$\begin{aligned} AF &= AB + BF \\ &= 1 + \frac{uv}{c^2} \end{aligned}$$

And the length of EF is

$$\begin{aligned} EF &= ED + DF \\ &= \frac{v + u}{c} \end{aligned}$$

And thus the speed of observer O'' relative to observer O is

$$\begin{aligned} \frac{w}{c} &= \tan \angle EAF \\ &= \frac{EF}{AF} \\ &= \frac{1}{c} \cdot \frac{u + v}{1 + \frac{uv}{c^2}} \\ w &= \frac{u + v}{1 + \frac{uv}{c^2}} \end{aligned}$$

□

Problems

Problem 1 Suppose Alice, Bob and Charlie live on three different planets which happen to be aligned. Alice's planet is 2 light hours to the right of Bob's planet, while Bob's planet is 1 light hour to the right of Charlie's planet. One day ($t = 0$), Charlie leaves his planet at a constant speed $0.6c$ to visit Alice. Once he arrives, Charlie sends a light signal to invite Bob to come over. However, it happens that Bob took off on his spaceship to the left at a speed $0.5c$ after 2 hours Charlie had left.

- When will Bob receive the light signal?
- Suppose that once Bob received the signal, he turns around and travels towards Alice at a speed $0.8c$. When will Bob arrive at Alice's planet?
- Draw the spacetime diagram of the entire event. Let ct be the y -axis and the position be the x -axis.

Notice that the light signal in the spacetime diagram should be at a 45 degree angle about the two axes.

Problem 2 Suppose Superman is trying to get from one end of a galaxy to another. The galaxy is 100 light years in diameter, and superman flies at a speed $0.8c$ relative to the space around him. At the end of the galaxy, there is this mysterious medium that flows in the same direction as superman is going. The medium is 20 light years across and flows at a speed $0.4c$. Thus, if superman is inside the medium, he defies Special Relativity and flies at a net speed $1.2c$. On the other hand, superman needs to sleep. In fact, at some point before he arrives at the end of the galaxy, he needs to spend 10 year sleeping. He wants to arrive at the other end of the galaxy as soon as possible, and thus has to decide whether it is better for him to sleep before getting into the medium or during.

- Draw the space time diagrams to illustrate the two different choices Superman has.
- What should Superman do?

Problem 3 As explained in the introduction, many years before Einstein, Galileo introduced the relativity principle (the first axiom or postulate). However the constancy of the speed of light was replaced by the assumption that the time is universal for all observers. Now, consider a frame at rest with coordinates (t, x) , and another frame (t', x') moving to the right with a velocity v with respect to the first. Based in your intuition and Galileo's assumption write down the equations that relate the coordinates in both frames. These are called

Galilean transformations. Now, using this equations prove that the relation

$$\frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{\Delta t} - v$$

holds between the velocities of an object in the first frame $\frac{\Delta x}{\Delta t}$ and in the other $\frac{\Delta x'}{\Delta t'}$.

Problem 4 Consider now the Lorentz transformations above, and rederive the velocity addition formula in special relativity of Example 1. As a check, try to observe that an object moving at the speed of light, c , in one frame, also moves at the same speed in the other frame. Similarly check that if one observer is at rest and sees the other moving with a velocity v , the other in its rest frame sees the first moving at velocity $-v$.

Problem 5 The *spacetime interval* Δs for a given observer with coordinates (x, t) is defined as

$$\Delta s^2 = -c^2(\Delta t)^2 + (\Delta x)^2$$

where Δt and Δx are the difference in the coordinates of two events. Check that general Lorentz transformations leave the spacetime interval invariant, i.e., $\Delta s'^2 = \Delta s^2$

Problem 6 Now that you've had some practice solving some problems using Lorentz transformations, let's try to derive them. Before you do, we highly suggest going back and reviewing the axioms of special relativity, and the conditions they impose on the form of Lorentz transformations. The following derivation is a direct continuation of that section.

- a. Consider how each Alice and Bob record Bob's motion:
 - i. Find an expression $x_B(t)$ for Bob's position as a function of time, as seen by Alice.
 - ii. Find an expression $x'_B(t')$ for Bob's position as a function of time, as seen by Bob.
 - iii. Use the general formula for x' (Eq 1) along with the above expressions for $x(t)$ and $x'(t)$ to solve for b in terms of a .

- b.** Now consider how each Alice and Bob record Alice's motion:
- i.** Find an expression $x'(t')$ for Alice's position as a function of time, as seen by Bob. 'Sanity check' – should this position be positive or negative?
 - ii.** Find an expression $x(t)$ for Alice's position as a function of time, as seen by Alice. Now write this as $x(t')$, a function of Bob's time.
 - iii.** Start with the expression in part (i). On the next line, replace x' and t' using (Eqs 1 and 2). Simplify using the result of part (ii), and solve for e in terms of a .

- c. Now imagine Alice emits a light ray (from her origin) in the $+x$ direction at $t = t' = 0$. Repeat part (b), this time using the light ray's position instead of Alice's. You should obtain an equation involving only constants (any of a, b, d, e, c, v) and t . Solve this equation for d in terms of the other constants. You should now be able to write b, d, e in terms of a .

- d.** Finally, we will exploit the symmetry between frames A and B . Previously, we started with the coordinates (x, t) in frame A and used a Lorentz transformation to obtain (x', t') in frame B . However, there's no reason why we can't start with the primed coordinates in frame B and use a Lorentz transformation to obtain the unprimed coordinates in frame A . That is, we have (using the results of the previous parts to constrain the form of the Lorentz transformation)

$$x = a(x' - v't') \tag{3}$$

$$ct = a\left(ct' - \frac{v'}{c}x'\right) \tag{4}$$

- i.** Notice that we used v' rather than v in Eqs 3 and 4. Find v' in terms of v .
- ii.** Eliminate x' from Eqs 3 and 4 and solve for t' as a function of x, t and a, c, v .
- iii.** Solve for a in terms of v and c by demanding the result of the previous part agree with Eq 2.

e. (Alternate Derivation) The preceding derivation of the Lorentz transformations used similar reasoning to Einstein's original derivation. However, modern physicists usually take a slightly different approach – one that elevates the invariance of the spacetime interval, Δs , from a derived fact to an axiom. Whereas Einstein's approach was physically-oriented, this interval-centric approach is more geometrically-oriented (and, in more advanced treatments, more amenable to group theory, which is the natural mathematical language for describing symmetry transformations).

- i. Rather than performing parts (c) and (d), use the requirement that $\Delta s = \Delta s'$ to constrain the result of parts (a) and (b).

Since we arrive at the same answer for the most general (linear) transformation between inertial frames using the interval as an axiom, we conclude that the the two sets of axioms are equivalent.