QUATERNIONS

LAMC HIGH SCHOOL 1

1. Complex Numbers

A complex number is anything of the form $a + bi$, where a, b are both real numbers. The sum and product of complex numbers are defined as though they are polynomials, except that $i^2 = -1$. The set of all complex numbers is denoted $\mathbb{C}$.

Problem 1: Show that if z is a nonzero complex number, so is $1/z$.

The **norm** of a complex number $a + bi$ is defined as $|a + bi| = \sqrt{a^2 + b^2}$.

Problem 2: Explain why the norm of a complex number $a + bi$ is its length when plotted as (a, b) in the plane.

Problem 3: The complex numbers with norm equal to 1 form a circle in the plane; what happens when you multiply two of them?

You can identify each complex number with a matrix, $\phi(a+bi) = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$

Problem 4: Show that the set of all such matrices really are the same as the complex numbers by checking the following:

(1) $\phi(1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
(2) $c\phi(a+bi) = \phi(c(a+bi))$
(3) $\phi(a+bi) + \phi(c+di) = \phi((a+bi)+(c+di))$
(4) $\phi(a+bi)\phi(c+di) = \phi((a+bi)(c+di))$
(5) The function ϕ from complex numbers to 2 by 2 matrices is one-to-one

2. The Quaternions

The complex numbers are a way to add and multiply pairs of real numbers such that the distributive property holds; the operations are commutative, associative, and multiplicative inverses exist for nonzero complex numbers. Satisfying these properties means that the complex numbers are a *commutative division algebra*

Definition 1 The set $\mathbb{R}^n$ with a multiplication function $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and with addition defined component-wise, is an **$\mathbb{R}$-algebra**, or real algebra, if the distributive laws hold:
((ax+y),z)=am(x,z)+bm(y,z)
m(x,(ay+bz))=am(x,y)+bm(x,z)
For any $a, b \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^n$.

Definition 2 A real algebra $\mathcal{A}$ is a division algebra if, given $a, b \in \mathcal{A}$ with $a \neq 0$, the equations $aX = b$ and $Xa = b$ have unique solutions in $\mathcal{A}$

Problem 5 Which of the following are $\mathbb{R}$-algebras? Which are division algebras?

(1) $\mathbb{R}$
(2) $\mathbb{C}$
(3) $\mathbb{R}^3$, with $m((a,b,c),(x,y,z)) = (ax, by, cz)$
(4) $\mathrm{Mat}(2,\mathbb{R})$, the set of 2×2 matrices with real entries.
(5) $\mathrm{Mat}(2,\mathbb{C})$, the set of 2×2 matrices with complex entries.

2.1. **A model for the Quaternions.** Let's define a multiplication on $\mathbb{R}^4$: write the element (a,b,c,d) as $a + bi + cj + dk$, and state that

$$i^2 = j^2 = k^2 = ijk = -1$$

Problem 6 Using the above equalities, and associativity of multiplication (but NOT commutativity), calculate:

(1) jki
(2) kij
(3) ij
(4) ji
(5) All remaining 2-term products of i, j and k (You don't need any more calculations to do this!)
(6) (a+bi+cj+dk)(a'+b'i+c'j+d'k)

Conclude that $\mathbb{R}^4$ with this multiplication (called the Quaternions, $\mathbb{H}$) is an $\mathbb{R}$-algebra.

Problem 7 If a, b, c, d aren't all zero, calculate what quaternion is $1/(a + bi + cj + dk)$, thereby showing that $\mathbb{H}$ is a division algebra.

The Quaternions can be viewed as living inside $\mathrm{Mat}(2,\mathbb{C})$; let

$E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $I = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $K = \left(\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}\right)$

Then $\Phi : \mathbb{H} \to \mathrm{Mat}(2,\mathbb{C})$ is defined as $\Phi(a + bi + cj + dk) = aE + bI + cJ + dK$

Problem 8 Show that the set of matrices of the form $aE + bI + cJ + dK$ are the same as the quaternions, by checking that $I^2 = J^2 = K^2 = IJK = -E$.

Problem 9 Verify that for $x = a + bi + cj + dk$, $x^2 = 2ax - (a^2 + b^2 + c^2 + d^2)$

Problem 10 Suppose b, c, d are real numbers such that $b^2 + c^2 + d^2 = 1$. Show that $(bi + cj + dk)^2 = -1$; give a geometric description of this set of "square roots of -1."

Problem 11 The set $Q = \{1, -1, i, -i, j, -j, k, -k\}$ is called the Quaternion group. What do you get if you quotient it out by the subgroup $\{1, -1\}$?

Problem 12 Can you find any real division algebras other than $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$?