SPHERICAL GEOMETRY (UCLA MATH CIRCLE)

1. Maps and areas

Mercator projection was invented by the Flemish geographer and cartographer Gerardus Mercator, in 1569. It can be understood from the following Figure:

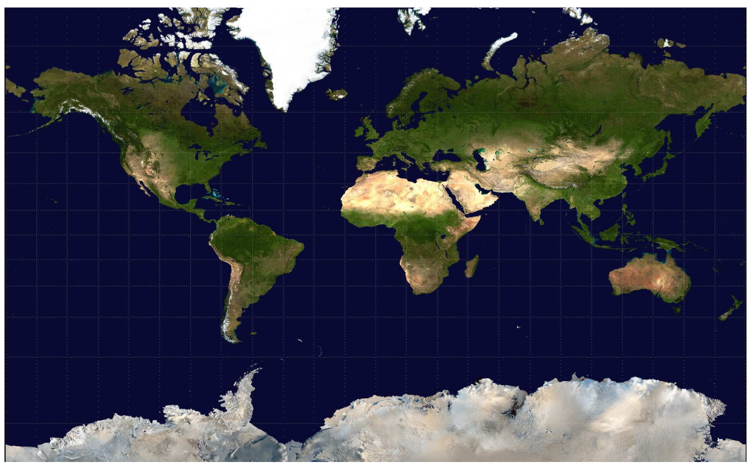

Figure 1. World map using Mercator projection (image from Wikipedia).

Lambert projection was invented by Alsatian mathematician Johann Heinrich Lambert in 1772. It can be understood from the following Figure:

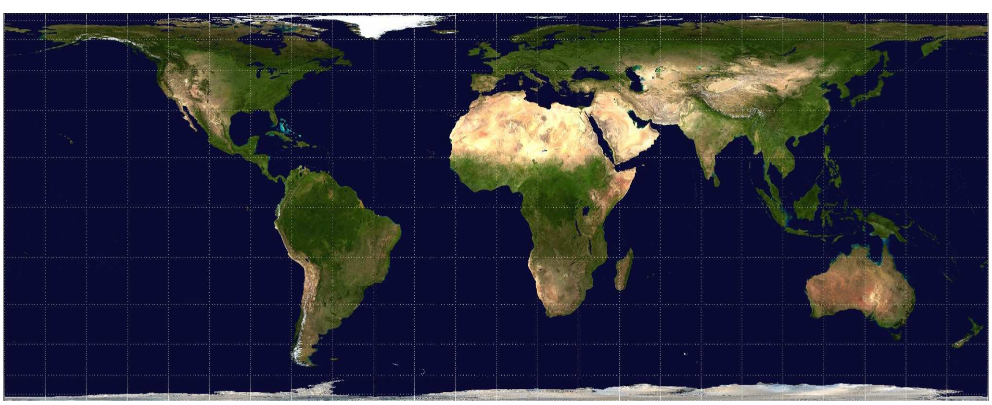

Figure 2. World map using Lambert projection (image from Wikipedia).

Exercise 1. Approximate the area ratios using each map (write down both answers):

a) area(Australia)/area(Greenland)

b) area(USA)/area(Australia)

c) area(North America)/area(Africa)

d) area(North America)/area(South America)

d) area(North + South America)/area(Earth)

Exercise 2. Use the following true area values to compute the ratios in the previous exercise:

- area(Africa) = 11.7 mln sq mi
- area(Australia) = 3 mln sq mi
- area(Greenland) = 0.8 mln sq mi
- area(Earth) = 196 mln sq. mi
- area(North America) = 9.5 mln sq mi
- area(South America) = 6.9 mln sq mi
- area(USA) = 3.8 mln sq mi

Compare the true ratios to those you found in Exercise 1). Conclude which map does a better job approximating the area.

Exercise 3. Assume now that Lambert projection always preserves the area ratios. Compute:

a) area(Arctic region)/area(Northern Hemisphere), where *Arctic region* is defined as everything above 66°. *Hint:* you can use the fact that $\sin(66°) \approx 0.91$

b) area(Tropics)/area(Earth), where *Tropics* are defined as everything between 23° latitude and −23° latitude. *Hint:* you can use the fact that $\sin(23°) \approx 0.40$

c) area(everything north of Tropics and south of Arctic)/area(Northern Hemisphere).

Exercise 4. Approximate radius of the Earth (*Hint*: use the surface area above and $\text{area} = 4\pi\ \text{radius}^2$ formula)

Exercise 5. Denote by N_ℓ the region on the surface of the earth at distance at most ℓ. Think of N_ℓ as "your neighborhood". Approximate the following ratios (in percentages, please!)

a) area($N_{10\,\text{mi}}$)/area(Earth). *Hint*: it's ok to use the usual formula for the area of a circle in this case.

b) area($N_{100\,\text{mi}}$)/area(Earth). *Hint*: still ok to use the usual formula for the area of a circle in this case.

c) area($N_{4{,}000\,\text{mi}}$)/area(Earth). *Hint*: it's NOT ok to use the usual formula for the area of a circle. Use the Lambert projection.

d) area($N_{1{,}000\,\text{mi}}$)/area(Earth). Use both methods and compare the results.

2. Kissing numbers

Exercise 6. (Kissing number in the plane)

Find the maximal number of nonoverlapping unit circles which can touch a fixed unit circle (as in the Figure 5). In *discrete geometry*, such circles are called *kissing*.

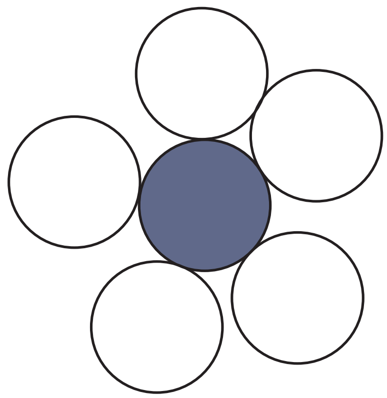

Figure 3. Five kissing unit circles.

Exercise 7. (Kissing number in the space, lower bound)

a) Prove that the there exist 12 nonoverlapping unit spheres which touch a unit sphere. *Hint*: think of oranges in the grocery store and how they fit together.

b) Show that one can place spheres in the vertices of an icosahedron. *Hint:* you can use the fact (discovered by Euclid) that the radius of a circumscribed sphere is $\approx 0.951 \cdot a$, where a is the length of an edge.

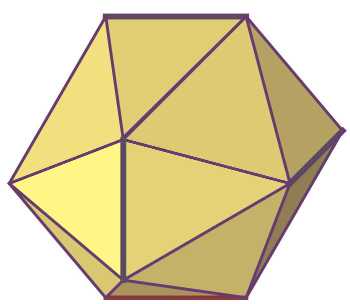

Figure 4. Icosahedron.

Exercise 7. (Kissing number in the space, upper bound)

a) Consider two touching spheres and the cone from the center of one over the other. Compute the height of the shaded circular cap U (*Hint*: use the idea on the right side of the Figure).

b) Use the Lambert projection once again to conclude that area$(U) \approx 4\pi/14.9283$

c) Deduce from there that there is no kissing configuration of 15 spheres.

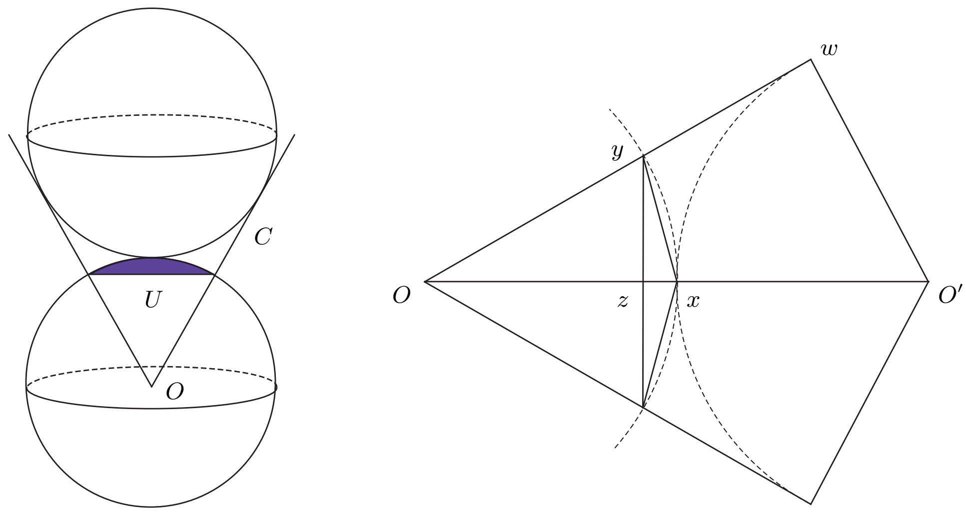

FIGURE 5. Computing the height of a cap U on a sphere.