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Warm-up

Problem 1 *Draw a room that cannot be completely lit by one light source in the shape of a point, but can be completely lit by two such sources.*

Problem 2 *Given a pile (more than two) of rectangular bricks,*

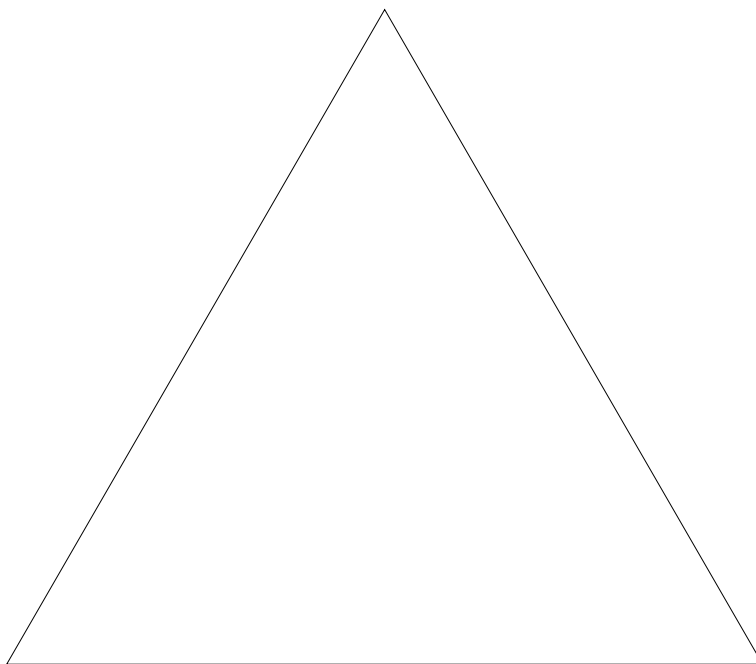


all of the same size, how can you use a ruler long enough to measure directly the length of the diagonal of a brick from the pile? “Directly” means without employing the Pythagoras’ Theorem. Please draw a picture to illustrate your solution.

Problem 3 Given $4^x + 4^{-x} = 23$, find $2^x + 2^{-x}$.

Problem 4 Let $a, b > 0$ and $a, b \neq 1$.
Prove that $\log_a b = 1/\log_b a$.

Problem 5 *Recall the construction of the Sierpinski triangle and use the frame below to draw the figure.*



Problem 6 *Find the dimension of the Sierpinski triangle.*

Decimal and rational representation of fractions

The purpose of this micro-course is to learn going back and forth between rational and decimal representations of fractions and to prove the following two theorems.

Theorem 1 *If p and q are co-prime integers, then the fraction p/q can be represented either in the terminating (finite) or in the infinite periodic (a.k.a. repeating or recurring) decimal form.*

Example 1 $\frac{2}{5} = 0.4$ $\frac{2}{7} = 0.285714285714\dots = 0.\overline{285714}$

Theorem 2 *A real number in the terminating (finite) or repeating (infinite periodic) decimal form can be represented as a fraction p/q where the integers p and q are co-prime.*

Example 2 $0.75 = \frac{3}{4}$ $0.757575\dots = 0.\overline{75} = \frac{25}{33}$

Problem 7 *Find a decimal expression for the fraction $2/9$.*

Problem 8 Convert the decimal $2.343434\dots = 2.\overline{34}$ to the rational form p/q , the integers p and q co-prime. If you can't do it right away, please get back to the problem after studying the theory on pages 8 – 11.

Problem 9 *What is the maximal possible length of the repeating part of the decimal representation of the fraction p/q , p and q co-prime integers? Hint: take a good look at the second part of Example 1.*

Problem 10 *Prove Theorem 1.*

The following sum

$$a + ax + ax^2 + ax^3 + \dots = \sum_{n=0}^{\infty} ax^n \quad (1)$$

is called *geometric series*. Let us consider the sequence of its partial sums, $S_0 = a$, $S_1 = a + ax$, $S_2 = a + ax + ax^2$, ... $S_n = a + ax + ax^2 + \dots + ax^n$.

Problem 11 *Prove the following formula.*

$$S_n = a \frac{x^{n+1} - 1}{x - 1} \quad (2)$$

Recall that the mathematical sentence $\lim_{n \rightarrow \infty} S_n = S$ means that for any real number $\epsilon > 0$, no matter how tiny, there exists a natural number N (that depends on ϵ) such that for any natural number $n > N$, $|S - S_n| < \epsilon$.

Problem 12 *Prove that for $|x| < 1$, the following is true.*

$$\lim_{n \rightarrow \infty} S_n = S = \frac{a}{1 - x} \quad (3)$$

Recall that the mathematical sentence $\lim_{n \rightarrow \infty} S_n = \infty$ means that for any real positive number M , no matter how huge, there exists a natural number N (that depends on M) such that for any natural number $n > N$, $|S_n| > M$.

Problem 13 *Prove that if $x = 1$ or $|x| > 1$ then the following is true.*

$$\lim_{n \rightarrow \infty} S_n = S = \infty \tag{4}$$

Problem 14 *Compute the sums S_0 through S_5 for $x = -1$. Find the general formula for S_n in this case. Does the $\lim_{n \rightarrow \infty} S_n$ exist for this value of x ?*

The above can be summarized in one statement. The geometric series $\sum_{n=0}^{\infty} ax^n$ converges to $a/(1-x)$ for $|x| < 1$ and diverges otherwise. This gives us a powerful tool for converting fractions from the recurring decimal to the rational form. Let us take another look at the second part of Example 2.

$$0.\overline{75} = \frac{75}{100} + \frac{75}{100^2} + \frac{75}{100^3} + \dots = \frac{3}{4} \left(1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \dots \right)$$

The sum in parenthesis is a geometric series with $a = 1$ and $x = 1/100$. Therefore,

$$0.\overline{75} = \frac{3}{4} \frac{1}{1 - \frac{1}{100}} = \frac{3}{4} \frac{1}{\frac{99}{100}} = \frac{3}{4} \frac{100}{99} = \frac{25}{33}.$$

Problem 15 *Convert the following to the rational form.*

- $1.\overline{1} =$

- $54.3\overline{21} =$

The problem continues to the next page.

- $0.06\overline{101} =$

- $0.\overline{9} =$

Problem 16 *Convert the following to the decimal form.*

- $\frac{7}{125} =$

- $\frac{11}{16} =$

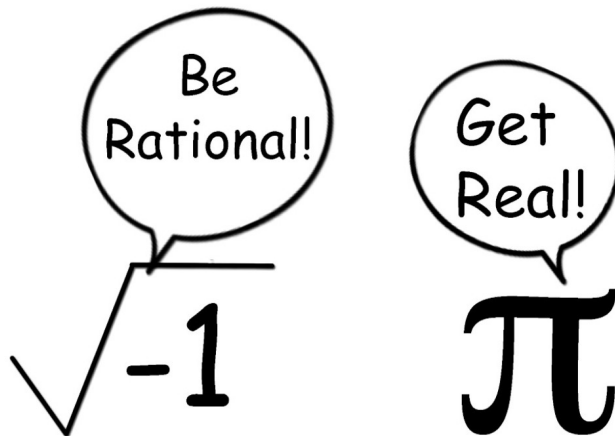
The problem continues to the next page.

- $\frac{11}{17} =$

- $6\frac{2}{35} =$

Problem 17 What fractions p/q , p and q co-prime integers, can be converted to the finite decimal form? Why?

A real number is called *rational*, if it can be represented as a fraction p/q where p and q are co-prime integers. A real number that is not rational is called *irrational*.



Problem 18 *Give an example of an irrational number. Explain why the number you have chosen cannot be rational.*

Problem 19 *Prove that $\sqrt{2}$ cannot be represented as a terminating or recurring decimal.*

Problem 20 *Prove Theorem 2.*

If you are finished doing everything above, but there still remains some time...

There has recently opened a middle/high school in San Francisco, called the *Proof School*, that has a potential to become the best math-oriented school in the US. They are currently hiring math teachers. Solving the following problem is a part of the selection process.

Problem 21 *Given four points, A , B , C , and D , in the plane such that no three of them are collinear, how many angles are determined by these points? If we measured all the angles and recorded the smallest angle measure, how large could this value be?*