# Golden Slugs

You are in a shop that sells golden slugs precisely calibrated to have whatever mass you choose. Back home at the salt mines you have a balance scale, and you need to determine the mass of various pieces of salt by putting slugs on one side of the balance, and a piece of salt on the other side.

0. What is the smallest number of slugs necessary to be able to balance any whole number of kilograms between 0 and 100? Keep in mind that you can choose the slugs to be whatever weights you would like?

   (Hint: When a problem involves astronomically large numbers (like 100), it sometimes makes the problem easier if you try it with smaller numbers first!)

* What is the smallest number of slugs needed if you are allowed to put golden slugs on both sides of the balance? (That is, you can try to balance some slugs vs. a piece of salt and some other slugs.)

! Can you prove that your answers are correct? This is to say that, if you think the answer is 15, for example, then you need to prove that it is possible to perform the task with 15 slugs, and *impossible* to do it with 14 or fewer slugs.

# What does decimal notation mean?

When we say that $x = 6503$ we mean that

$$x = 6 \times 10^3 + 5 \times 10^2 + 0 \times 10^1 + 3 \times 10^0.$$

This is what is known as *positional notation*: the value of each digit in the decimal notation for $x$ varies with the position of the digit:

$$\begin{array}{cccc} 6 & 5 & 0 & 3 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 10^3 & 10^2 & 10^1 & 10^0 \end{array}$$

Thus the value of each digit increases by a factor of 10 as we move right to left. Very soon we'll use a different system of notation in which the role of 10 is played by a different number. To indicate that we are dealing with the decimal notation, we'll write $x = (6503)_{10}$.

# What does binary notation mean?

In binary (or "base 2") notation, only the digits 0 and 1 are used. As in decimal notation, each digit again has a different value depending on its position. This value increases by a factor of 2 (rather than 10) when we move to the left. We'll use the subscript 2 to indicate binary notation.

To say that $N = (1101)_2$ means that

$$N = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0.$$

Thus

$$N = 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 = 13.$$

The value of each digit is indicated below:

$$\begin{array}{cccc} 1 & 1 & 0 & 1 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 2^3 & 2^2 & 2^1 & 2^0 \end{array}$$

- Fill in the following table:

| Binary notation for $N$ | Expression for $N$ | Value of $N$ |
|---|---|---|
| 1101 | $1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$ | 13 |
| 10 | | |
| 100 | | |
| 1111 | | |

The table below lists various powers of 2 and may come in handy:

| $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | $2^7$ | $2^8$ | $2^9$ | $2^{10}$ | $2^{11}$ | $2^{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |

# Finding binary notation for a number

## How many binary digits does a number $N$ require?

Given $N$, find the largest power of 2 that fits into $N$ (i.e., is no larger than $N$). For example, if $N = 11$, the largest power of 2 that fits into $N$ is $8 = 2^3$. The number of digits required is then one more than this power (4 in this case).

- Fill in the following table:

| $N$ | Largest power of 2 that fits into $N$ | Number of binary digits |
|---|---|---|
| 11 | $8 = 2^3 \leq 11$ | 4 |
| 9 | | |
| 13 | | |
| 15 | | |

## Finding the binary notation for a number $N$

Now that we know how many digits $N$ has we know the left-most digit (it's 1, the only nonzero digit we have). For example, we therefore know that $11 = (1???)_2$ since 9 requires 4 digits. Now subtract from $N$ the largest power of 2 that fits: $11 - 2^3 = 11 - 8 = 3$. Find the largest power of two that fits into that number ($2^1 = 2$ fits into 3). This gives you that the next non-zero digit is the one corresponding to $2^1$: $11 = (101?)_2$. Now subtract from 3 the largest power of two that fits into it ($3 - 2^1 = 3 - 2 = 1$) and find the largest power of 2 that fits into that number ($2^0 = 1$). This gives you that the next nonzero digit is the one corresponding to $2^0$: $11 = (1011)_2$.

Fill in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 11 | $8 = 2^3$ | $11 - 8 = 3$ | $2 = 2^1$ | $3 - 2 = 1$ | $1 = 2^0$ | — | — | $\begin{array}{cccc} 1 & 0 & 1 & 1 \\ \uparrow & & \uparrow & \uparrow \\ 2^3 & & 2^1 & 2^0 \end{array}$ |
| 13 | | | | | | | | |
| 15 | | | | | | | | |
| 9 | | | | | | | | |

# Binary Problems

1. (Practice) Fill in the tables below to practice converting between base 10 and binary:

| Base 10 | Base 2 |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 12 | |
| 24 | |
| 48 | |
| 96 | |
| 32 | |
| 1024 | |
| 2048 | |
| 225 | |
| 450 | |

| Base 2 | Base 10 |
|---|---|
| 1 | |
| 10 | |
| 101 | |
| 110 | |
| 111 | |
| 111 | |
| 11101 | |
| 11011 | |
| 11111 | |
| 111110 | |
| 1111100 | |
| 1010101 | |
| 10000000000 | |
| 100000000000 | |

10. When written in binary, how many zeros are there at the end of ...

| $N$ | Trailing 0's in binary |
|---|---|
| 24 | |
| $24^{24}$ | |
| 25! | |
| $1234321^2 - 1$ | |
| $(100!) \cdot (200!)$ | |

11. Carry out the following subtraction problems in binary:

| | | | | | |
|---|---|---|---|---|---|
| $11_2$ | $10_2$ | $101_2$ | $10011_2$ | $10011_2$ | $1000000_2$ |
| $-1_2$ | $-1_2$ | $-11_2$ | $-1001_2$ | $-111_2$ | $-\ 1_2$ |

100. Indicate which of the following numbers is divisible by 3:
(a) $1_2$ (b) $11_2$ (c) $110_2$ (d) $10101_2$ (e) $1101110101_2$
(f) $10101011110101111_2$
(g) $110101010011110101010101010110101011111110101011111_2$
(h) $11010100000101010011111111101000100001001011101110101001000101011010101011_2$
What is a general rule for determining whether a number is divisible by 3 based on its binary representation?

101.* How might you represent the fraction $\frac{1}{3}$ in binary?