Math Circle
Beginners Group
January 10, 2016
Geometry I
Solutions

Warm-up Problem

A bridge is on the verge of collapse. Four people want to cross the bridge to reach a safer area before it collapses. It is a dark, stormy night, and there is only one flashlight available among them. The flashlight is required to make a trip across the bridge. The bridge can only take the weight of two people at a time. Alan takes a minute to cross the bridge. Ben takes 2 minutes. Cathy takes 5 minutes. Das takes 10 minutes to cross the bridge. What is the shortest time the four people will take to cross the bridge, and how?

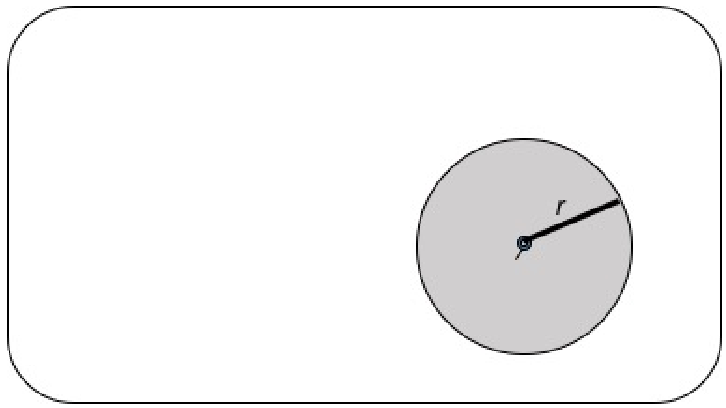
> This is the sequence the four people should follow.
>
> Alan and Ben will first cross the bridge with the flashlight, taking 2 minutes. Alan will then come back with the flashlight taking 1 minute.
>
> Cathy and Das will cross next, taking 10 minutes, and they will pass the flashlight to Ben who is already on the other side. Ben will then come back taking 2 minutes.
>
> Alan and Ben will finally cross the bridge together, taking 2 minutes.
>
> Now, all four people are on the other side, and they took 17 minutes in total.

Tying Goats[1]

This problem set deals with goats. Goats are ravenous and consume everything they can reach. Therefore, they are usually kept on a rope attached to a stake planted in the pasture.

1. Imagine that in the field below, a goat is attached to the given stake with a rope of length r.

   (a) Draw the section of the pasture that can be consumed by the goat.

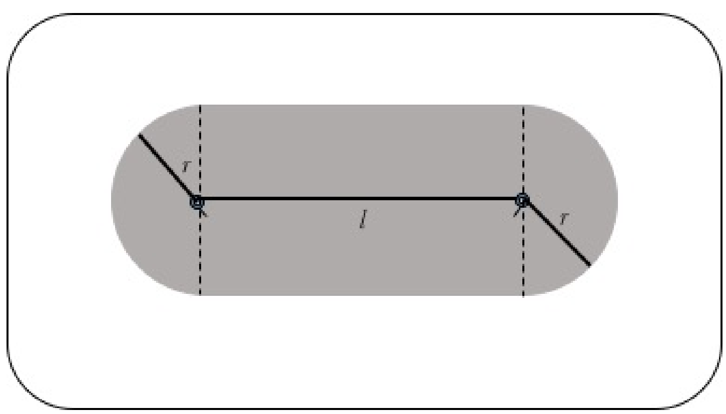


   (b) What is the area (in terms of r) of the section the goat can consume?

   The section is a circle with radius r, and area πr^2.

2. Now imagine that a rope of length l has been stretched between two stakes in a field. A goat is tied to this rope with another rope of length r using a ring that is free to slide along the first rope.

   (a) Draw the section of the pasture that can be consumed by the goat.



[1] The problems are taken from "A Moscow Math Circle: Week-by-week Problem Sets" by Sergey Dorichenko.

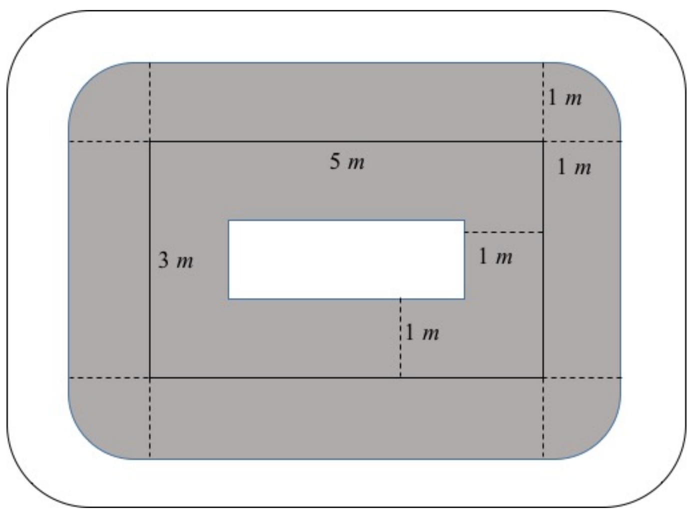
(b) What is the area (in terms of l and r) of the section the goat can consume?

About any point that lies on the line l, the goat can consume a section in the shape of a circle of radius r. Consecutive and equal-sized circles, which are co-situated horizontally, will form a stadium shape as shown above. The area of this stadium shape can be found by breaking the stadium shape into a rectangle and two semi-circles. The total area will be $l \cdot 2r + \pi r^2$.

These two shapes were simple to construct. So we can call the circle and the stadium shapes that you obtained above our "basic shapes." Let us try to use these shapes for more complicated setups of ropes.

3. A mathematician took a walk on a field holding a goat on a 1-meter-long rope. The mathematician's path was rectangular with dimensions 3 meters by 5 meters. Draw the section of the field the goat will have consumed by the end of the mathematician's walk.

(a) Draw the section of the pasture that can be consumed by the goat.



(b) What is the area, expressed in square meters, of the section the goat can consume?

This problem is an extention of the previous. There will be stadium shapes around each side of the rectangle that the mathematician walks. The area of the section can be found by breaking the grey portion into rectangles and quarter-circles. So, the total area will be $(4(1 \times 3 + 1 \times 5) - 4 \times 1) + \pi \cdot 1^2 = 28 + \pi$.

What if we want to constrain the area that the goat can consume to a semi-circle, or any other shape?

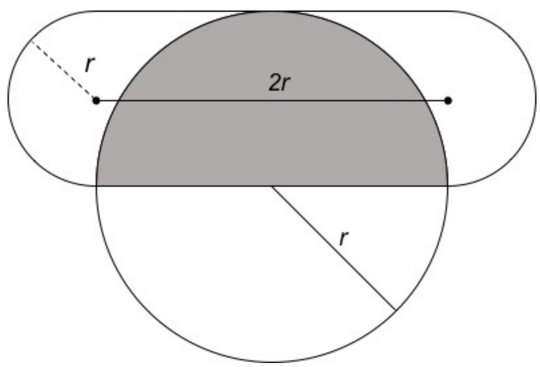
We would have to use more than one rope of different lengths. This way, we can have basic shapes of different sizes, i.e. circles of different radii and stadiums of different dimensions. The careful intersection of these basic shapes will yield the shape that we desire.

Suppose we know how to tie the goat so that it can only eat the grass in a particular basic shape X, and we also know how to tie the goat so that it can only eat the grass in a basic shape Y. If we want the goat to eat only the grass in the intersection of the shapes X and Y, we should tie the goat in both ways.

4. How can a goat be constrained to consume grass in the shape of

   (a) a semi-circle of radius r?

      i. How do two basic shapes intersect to form a semi-circle? Show the intersection below. What are the lengths of the ropes that you will use?

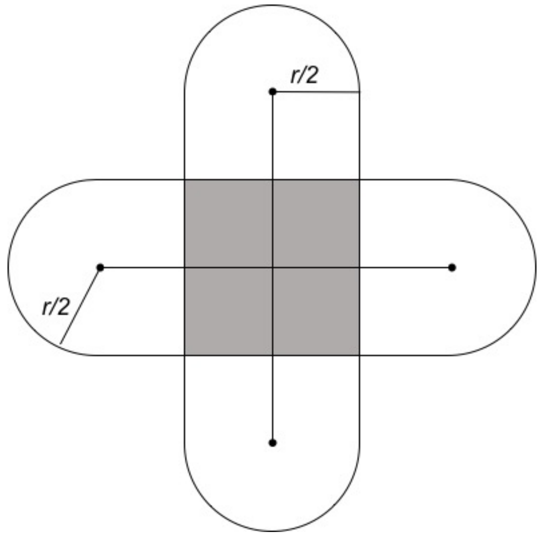


      ii. What is the area of the pasture that the goat can consume? The answer should be in terms of r.

         The goat will be tied two ways as shown above. It will be tied to a stake in the ground with a rope of length r (for the circular formation). It will also be tied to another rope of length $2r$ using rope of length r attached to a ring that is free to slide along the first rope – to make the shape of a stadium (or garlic bread). The area of the semi-circle formed by the intersection is $\frac{1}{2}\pi r^2$.

   (b) a square of side length r?

      i. How do two basic shapes intersect to form a square? Show the intersection below. What are the lengths of the ropes that you will use?

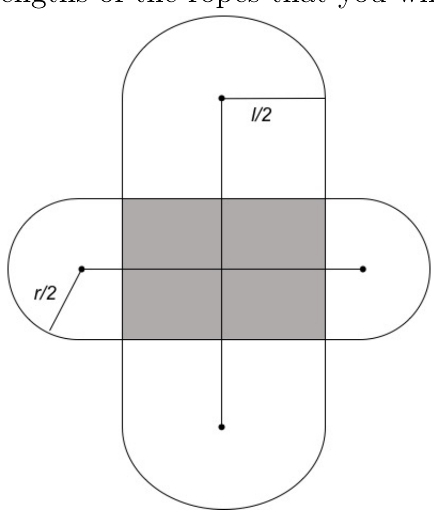


ii. What is the area of the pasture that the goat can consume? The answer should be in terms of r.

The goat will be tied two ways that result in two stadiums. The lengths of the ropes tied to the rings and the goat should be the same. The area of the intersecting square is r^2.

(c) a rectangle of length l and width r?

i. How do two basic shapes intersect to form a rectangle? Show the intersection below. What are the lengths of the ropes that you will use?

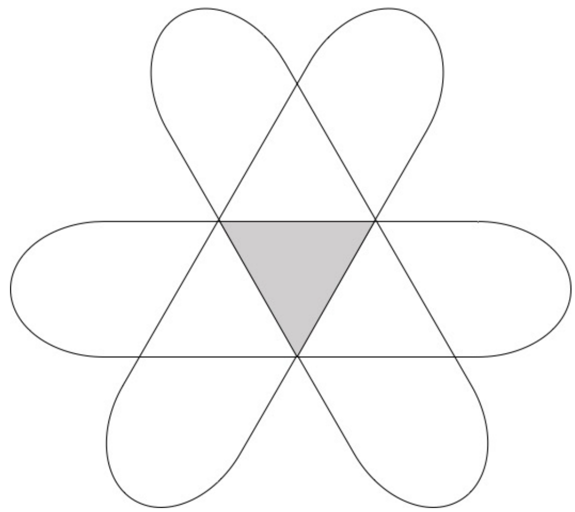


ii. What is the area of the pasture that the goat can consume? The answer should be in terms of l and r.

The area of the rectangle is lr.

(d) an equilateral triangle of side length r?

i. How do three basic shapes intersect to form a triangle? Show the intersection below. What are the lengths of the ropes that you will use?
You do not have to calculate the lengths of the ropes.



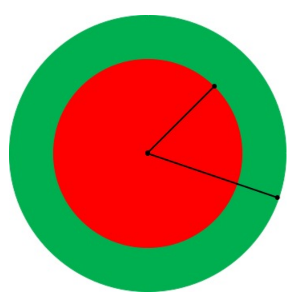
ii. What is the area of the pasture that the goat can consume?

You do not have to calculate the area.

5. Dogs can be used to herd goats because a goat will not occupy a space that a dog can reach. However, do not let a dog run free since it will chase the goat constantly and never let the goat rest or eat.

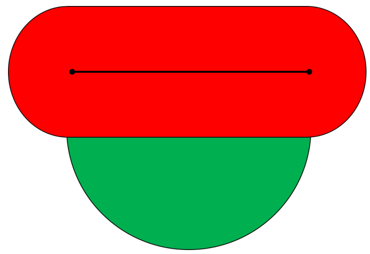
(a) How can a dog hold a goat in the shape of a ring? Draw the setup below.

We can tie the dog to the same stake as the goat, with the dog's rope shorter than the goat's.



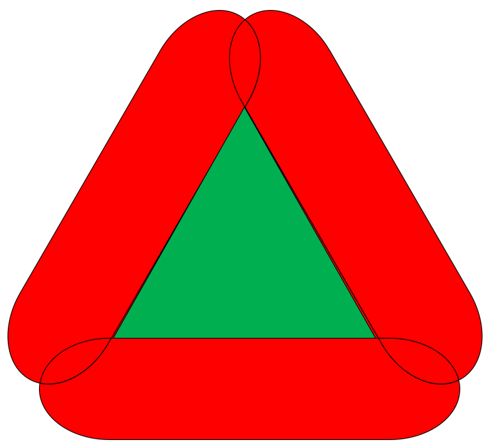
(a) How can a dog hold a goat in the shape of a semi-circle? Draw the setup below.

If the goat is tied to a rope staked in the ground, and the dog is tied to stay in a stadium with a straight side on a diameter of the goat's circle, then the goat will stay in the green semi-circle region that is inside the circle, but outside the stadium.



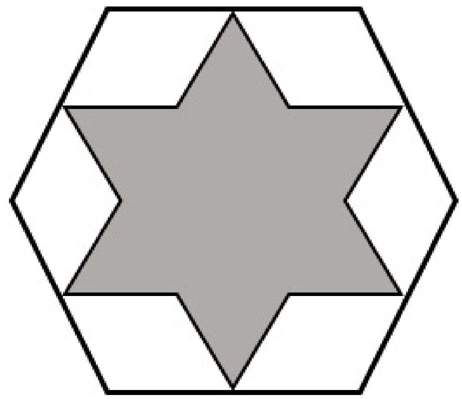
(a) Using tied dogs, contain an untied goat in the shape of a triangle. Draw the setup below.

There are three dogs, who are all tied in a way that each can traverse a red stadiums. An untied goat will thus move anywhere in the green triangular space.



Math Kangaroo[2]

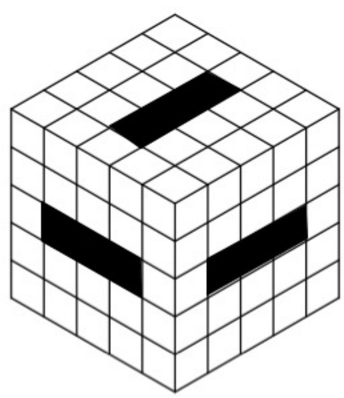
1. The vertices of the star shown in the picture are the midpoints of the sides of a regular hexagon. If the area of the star is 6, what is the area of the hexagon?



Break up the star into triangles, and find the area of each little triangle. Then break up the white portion of the hexagon into triangles of the same size. Find the area of the white portion. Then find the total area.

Answer: 12

2. The cube shown is made out of small cubes. Inside the big cube, tunnels were made going through the cube in such a way that they are parallel to the walls of the cube (see picture). After making the tunnels, how many small cubes are left in the solid?



[2]The problems are taken from Math Kangaroo contests from the years 2001, 2002, 2011 and 2013.

Total little cubes = 125

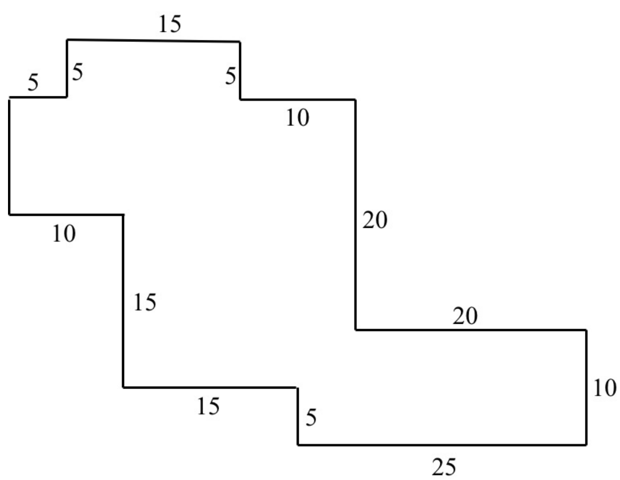
Imagine only one tunnel. The number of little cubes removed = 15

Imagine the second tunnel. Number of cubes removed, excluding those already counted = $15 - 3 = 12$

Imagine the third tunnel. Number of cubes removed, excluding those already counted = $15 - 3 - 3 = 6$

Answer: $125 - 15 - 12 - 6 = 92$

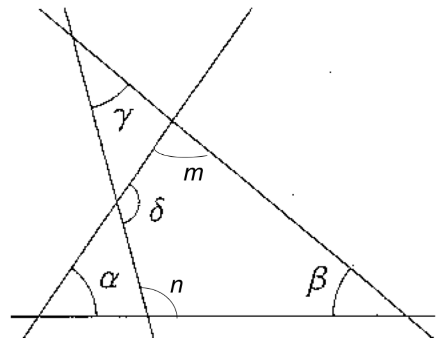
3. Janusz's garden is shaped in the way shown in the picture. The lengths of its sides are given in meters, and any two adjacent sides are perpendicular. What is the area of the garden in square meters?



Break up the figure into rectangles and find the area of each little rectangle.

Answer: $900m^2$

4. In the diagram, $\alpha = 55°$, $\beta = 40°$, and $\gamma = 35°$. What is the measure of δ?



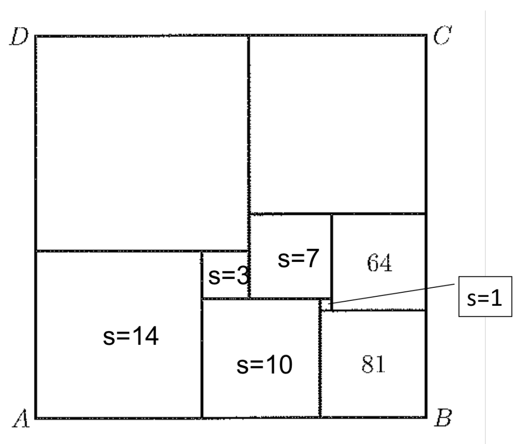
Use the angle-sum property of triangles to find $\angle m$ in the triangle with angles α and β.

Use the angle-sum property of triangles to find $\angle n$ in the triangle with angles γ and β.

Use the angle-sum property of quadrilaterals to find δ in the quadrilateral with angles m,n,β and δ.

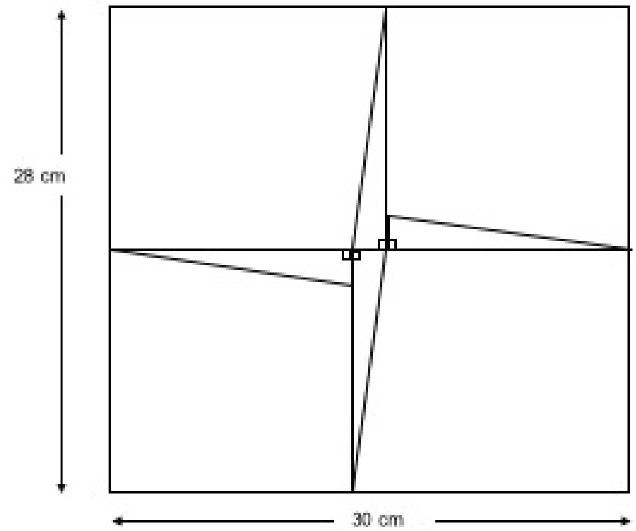
Answer: 130°

5. Rectangle $ABCD$ is divided into 9 squares. The areas of two of the squares are $64in^2$ and $81in^2$, as shown in the picture. What is the length of the side AB?



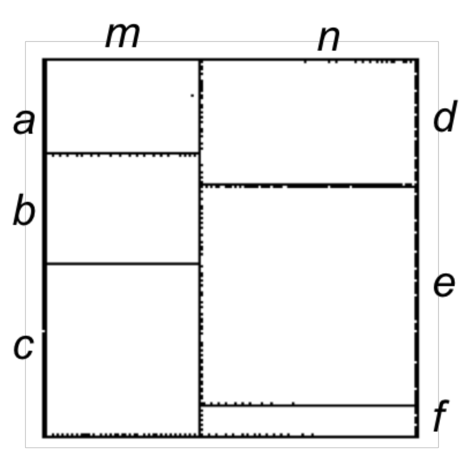
Use the given areas to find the side lengths of the little squares, and add them up to get $AB = 33in$.

6. There are four identical right triangles inside a rectangle, as shown in the picture. The lengths of the two sides of the rectangle are $28cm$ and $30cm$. What is the sum of the areas of all four triangles?



Since the side length is $28cm$ and all the triangles are equal in size, we know that the height of the triangles are $14cm$. This must mean that the base is $2cm$ in length because the other side of the rectamgle is $30cm$. So, area of all triangles is $4 \times \frac{1}{2} \times 14 \times 2 = 56cm^2$.

7. A square piece of paper was cut into six rectangles as shown. The sum of the perimeters of these six rectangles is equal to $120cm$. What is the area of the sheet of paper?



Label the pieces of the sides as above. Let the side of the square be s.

So, $a + b + c = d + e + f = m + n = s$.

We know that the perimeter is equal to $120cm$.

Therefore, $2(a + m + b + m + c + m + d + n + e + n + f + n) = 120$

$$\Rightarrow 2(a + b + c + d + e + f) + 6(m + n) = 120$$

$$\Rightarrow 2(2s) + 6s = 120$$

$$\Rightarrow 10s = 120$$

Therefore, $s = 12$, and the area of the square is $144cm^2$.