Pick the right modulus!

In each of the following problems, the key is to consider the numbers in the problem modulo m for some m, and do a case-by-case analysis. For some, the best modulus is given. For others, you have to guess! (But guess judiciously.)

1. If $p = n^5 + 4n$ is a prime number, find p. (Use $m = 5$.)
2. p, q, r are prime numbers greater than 3 which lie in an arithemtic progression. That is, $q = p + d$, $r = p + 2d$. Prove that d is divisible by 6. (Use $m = 6$.)
3. Prove that $n^3 - n$ is divisible by 24 for any odd n.
4. $a^2 + b^2 + c^2$ is divisible by 9. Prove that out of a, b, c, there are two numbers such that the difference of their squares is divisible by 9.
5. If p and $8p^2 + 1$ are prime numbers, find p.
6. If a, b, c are odd, can $a^2 + b^2 + c^2$ be a perfect square?

Proofs!

1. Prove that $\gcd(m, n) \cdot \text{lcm}(m, n) = mn$.
2. Prove that if $m = an + r$ for some integers a, r, with $0 \leq r < n$, then $\gcd(m, n) = \gcd(n, r)$.
3. (a) Prove that $d = \gcd(m, n)$ is the least positive integer linear combination of m, n.
   (b) Prove that a number y is an integer linear combination of m, n iff $d \mid y$.
4. Prove there exists a number n such that $n, n+1, n+2, \cdots, n+2009$ are all composite numbers.
5. Prove that if $(n-1)! + 1$ is divisible by n, then n is a prime number.