

# Invariants II

LA Math Circle (Advanced)

November 15, 2015

Recall that an invariant is some quantity associated with a system that is left unchanged by a specified process. We typically use them to show that no matter how long this process goes on, the system will never be in a certain state. Let's solve some problems!

**Problem 1** On a tropical island there are 20 red chameleons, 13 blue chameleons, and 10 green chameleons. When two chameleons of different color meet, they change to the third color. Can all the chameleons eventually be of the same color?

*Solution.* Yes, all the chameleons can eventually be the same color. Here's a way for all of them to become red: If a red chameleon and a blue chameleon meet, they each become green. So, there are now 19 red chameleons, 12 blue chameleons, and 12 green chameleons. Now the blue and green chameleons can each pair up, creating 24 more red chameleons, so that all the chameleons are red.

**Problem 2** Consider an  $8 \times 8$  chessboard with two opposite corners removed. Can you cover the chessboard with dominoes? (Hint: First color the chessboard like in real life. Find an invariant related to the colors of the squares.)

*Solution.* Assume that both of the squares removed are black (note: they must be the same color, look at a chessboard!). Then the invariant is the **number of white squares showing minus the number of black squares showing**. We must prove that this quantity is actually invariant. So, suppose at some stage of covering the board with dominoes, there are  $W$  white squares showing and  $B$  black squares showing. Then when we put down a domino, *the domino must cover one black square and one white square* no matter whether we put the domino down vertically or horizontally. Therefore,

after putting down a domino, there are  $W - 1$  white squares showing and  $B - 1$  black squares showing, and

$$(W - 1) - (B - 1) = W + (-1) + (-B) + 1 = (W - B) + (1 - 1) = W - B.$$

So, as we claimed, the quantity  $W - B$  is unchanged, or invariant, under the process of putting down a domino. Why does this show that it is impossible to cover the board with dominoes? Well, before we put down any dominoes, there were two more white squares than black squares since two black squares were removed. This means that at the beginning, we had  $W - B = 2$ . Since  $W - B$  is invariant under putting down dominoes, at the end of the process of covering up the board with dominoes, we must still have  $W - B = 2$ . But if we could completely color the board, there would be no white squares showing and no black squares showing, and so  $W - B = 0 - 0 = 0 \neq 2$ !

**Problem 3** Prove that a  $10 \times 10$  board cannot be covered by T-shaped tetrominos (see the picture).



*Solution.* This problem is quite similar to the last problem. Suppose that we could cover a  $10 \times 10$  board with tetrominos. We will try to find a contradiction to show that this is impossible. If we color the board like a chessboard, then putting a tetromino will either cover one black square and three white squares, or one white square and three black squares. It would take 25 tetrominos to cover the board (since  $4 \times 25 = 100$ ). Each tetromino covers an odd number of black squares (either 1 or 3). There are a total of 50 black squares on the board, so this means that we can write 50 as a sum of 25 1's and 3's (To be more precise, if  $T_1, T_2, \dots, T_{25}$  are the 25 tetrominos, and  $T_1$  covers  $B_1$  black squares,  $T_2$  covers  $B_2$  black squares, and so on, then  $50 = B_1 + B_2 + \dots + B_{25}$ , and each  $B_i$  is equal to either 1 or 3). But when you add up an odd number of odd numbers, the result is always odd. Since 50 is even, this is a contradiction.

**Problem 4** Let  $n$  be an odd natural number. Start with the numbers  $1, 2, \dots, 2n$  and perform the following operation: cross off any two of them, say they are  $a$  and  $b$ , and replace them with the number  $|a - b|$ . Do this until only one number remains. What is the parity of the last number?

*Solution.* The invariant is **the parity of the sum of all the numbers in the list**. Let's show that this is actually invariant. If at some point in this process, the numbers in the list are  $a, b, c_1, c_2, \dots, c_k$ , then after performing the operation, the numbers remaining in the list will be  $|a - b|, c_1, c_2, \dots, c_k$ , so we have to show that  $a + b + c_1 + c_2 + \dots + c_k$  and  $|a - b| + c_1 + c_2 + \dots + c_k$  have the same parity. To do this, we only have to show that  $a + b$  and  $|a - b|$  have the same parity, since  $c_1 + c_2 + \dots + c_k$  certainly has the same parity as itself! We leave this as an exercise. The idea is to split it the argument into four cases. The first is that  $a$  and  $b$  are both even. Then  $a + b$  is even and  $|a - b|$  is even. Can you do the other three cases?

Now since the parity of the sum of all the numbers is invariant, and the sum of a list of one numbers is just that number, the parity of the final number will be the same as the parity of the sum  $1 + 2 + 3 + \dots + 2n$ . We've seen several ways to show that  $1 + 2 + \dots + m = \frac{m(m+1)}{2}$  this year (the induction handouts and the proofs without words handouts). Plugging in  $m = 2n$ , we get  $1 + 2 + \dots + 2n = \frac{2n(2n+1)}{2} = n(2n + 1)$ . Moreover,  $n$  was assumed to be odd, and  $2n + 1$  is always odd, no matter what  $n$  is. Therefore, their product is odd. Since their product is  $1 + 2 + \dots + 2n$ , we see that the last number will always be odd.

**Problem 5** An  $n \times m$  table is filled with real numbers such that each row and column add up to 1. Prove that  $m = n$ .

*Solution.* The point is that if you compute all the row totals and add them up, it's the same as computing all the column totals and adding them up. In the first case, since all the row totals are 1, the result is simply the number of rows:  $n$ . In the second case, again all the column totals are 1, so the result is  $m$ . Therefore,  $n = m$ .

Why is the first sentence of the solution true? Associativity and commutativity of addition! For instance, in the table

$$a_{11}a_{12}a_{13}a_{14}$$

$$a_{21}a_{22}a_{23}a_{24}$$

$$a_{31}a_{32}a_{33}a_{34}$$

Adding up all the row totals gives  $(a_{11} + a_{12} + a_{13} + a_{14}) + (a_{21} + a_{22} + a_{23} + a_{24}) + (a_{31} + a_{32} + a_{33} + a_{34})$  while adding up all the column totals gives  $(a_{11} + a_{21} + a_{31}) + (a_{12} + a_{22} + a_{32}) + (a_{13} + a_{23} + a_{33}) + (a_{14} + a_{24} + a_{34})$ . Both are simply equal to the sum of all the numbers in the table! This observation (along with a few generalizations...) is called Fubini's Theorem.

**Problem 6** Begin with the number

9, 999, 333, 666, 369, 999, 999, 999, 999, 991, 999, 999, 333, 369.

If you add up all of its digits, then add up all the digits of the resulting number, and so on, will you ever get the number 12? (No credit for actually doing the computation, you must find an invariant! Hint: divisibility criterion.)

*Solution.* The divisibility by three criterion states that a number is divisible by 3 if and only if the sum of its digits is. This number is not divisible by 3 (Why? Use the divisibility by 3 criterion. There's a 1, and all the other digits are divisible by 3, so the sum of this number's digits is not divisible by 3). Therefore, if we add up the digits in the sum of the numbers digits, then repeat, and repeat... the divisibility by 3 criterion states that we will never get a number which is divisible by 3.

**Problem 7** A delicious cherry pie is divided into 6 equal slices. One fresh cherry is placed on top of each slice. Jeff wants to play with the cherries on top to try to put them all on his slice. His mom, a clever mathematician, will only allow this under the condition that Jeff can only shift any two cherries to slices bordering those they stand on at the moment (he must move them at the same time). He can do this as many times as he likes. Can he meet his goal? (Hint: start by numbering the slices 1 through 6. Try to find an invariant based on the numbers of the slices containing cherries.)

*Solution.* Label the cherries  $C_1, C_2, \dots, C_6$ . Let  $C_1$  be on the slice num-

bered  $S_1$ ,  $C_2$  be on slice  $S_2$ , and so on. Then the **parity of the sum**  $S_1 + S_2 + \cdots + S_6$  **is invariant**. To see this, simply note that moving one cherry to an adjacent slice changes the parity of the sum, so moving to cherries to adjacent slices changes the parity twice, so the parity stays the same. Since  $1 + 2 + \cdots + 6$  is odd, while 6 times any number is even, this shows that Jeff can never get all six of the cherries on one slice. (Why?)

**Problem 8** Can a  $10 \times 10$  chessboard be covered up with  $4 \times 1$  tiles? (Hint: to find an invariant, start by writing the numbers  $-1, 3, -1, -1, -1, 3, -1, -1, -1, 3$  in the first row of the chessboard; then the same sequence in the second row, but shifted one spot to the left so that it starts with 3, then the same sequence in the third row shifted one more spot to the left, etc.)

*Solution.* Once you've drawn out the chessboard described, you can see that putting down a  $4 \times 1$  tile does not change the **sum of all the numbers showing on the board**, since you must cover one 3 and three  $-1$ 's. Therefore, this sum is invariant. The sum is initially 4 (make sure I'm not lying to you), and if you could cover the board completely, the sum would have to become 0 (the sum of zero numbers is 0), which is impossible.

**Problem 9** Begin with the numbers 1, 2, and 4. Now take any two of the numbers  $a, b$ , cross them out, and replace them with the numbers  $.6a + .8b$  and  $.8a - .6b$ . Repeat this operation as many times as you like. Can you ever obtain the numbers 2, 3, 3?

*Solution.* Check your mastery of algebra by showing that  $(.6a + .8b)^2 + (.8a - .6b)^2 = a^2 + b^2$ . This means that **the sum of the squares of the three numbers** is unchanged by the replacement described above. Since  $1^2 + 2^2 + 4^2 \neq 2^2 + 3^2 + 3^2$ , you can never obtain the numbers 2, 3, 3.

Invariants can also help us to solve physical problems. Consider an object with mass  $m$ , moving at speed  $v$  and whose vertical distance from the ground is  $h$ . Let  $g = 9.8m/s^2$  be the acceleration of gravity. Then experiments have shown that the quantity

$$\frac{1}{2}mv^2 + mgh$$

does not change unless the object interacts with another object. The quantity

$\frac{1}{2}mv^2$  is called the object's *kinetic energy*, the quantity  $mgh$  is called the object's *potential energy*, and their sum is simply called the object's *energy*. Consequently, this law is called the Law of Conservation of Energy.

**Problem 10** Consider a roller coaster which starts from rest at 300 meters in the air, and which does not use a motor since it is unaffected by friction. The roller coaster then descends 100 meters, goes through three loopdy-loops, then a hairpin turn, then goes up a minor hill. At the top of this hill, its height is 140 meters. Use the law of conservation of energy to determine the speed of the coaster at the top of the hill.

The Conservation of energy says that if  $v_i, h_i$  are the initial speed and height of the coaster (at the beginning of the ride), and  $v_f, h_f$  are the final speed and height of the coaster (at the top of the hill), then

$$\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}mv_f^2 + mgh_f.$$

Since the cart starts from rest, its initial speed  $v_i$  is 0. It's initial height  $h_i$  is 300. It's final height  $h_f$  is 140, and we are trying to find  $v_f$ . Also, we know that  $g = 9.8$ . Plugging the known quantities into the equation above gives us

$$m(9.8)(300) = \frac{1}{2}mv_f^2 + m(9.8)(140).$$

Since roller coasters always have positive mass, we can divide both sides by  $m$ , giving us

$$9.8 \cdot 300 = \frac{1}{2} \cdot 9.8v_f^2 + 9.8 \cdot 140.$$

Now just solve the equation and use a calculator. You should get 17.9 meters per second.

**Problem 11** Prove that if every room in a house has an even number of doors, then there must be an even number of doors on the outside of the house.

*Solution.* Let the rooms be  $R_1, R_2, \dots, R_k$  and let room  $R_1$  have  $d_1$  doors, room  $R_2$  have  $d_2$  doors, etc. Then since  $d_1, d_2, \dots, d_k$  are all even, so is  $d_1 + d_2 + \dots + d_k$ . Let's take a moment to think about the sum  $d_1 + d_2 + \dots + d_k$ . Since all doors on the inside of the house are the doors of exactly two rooms,

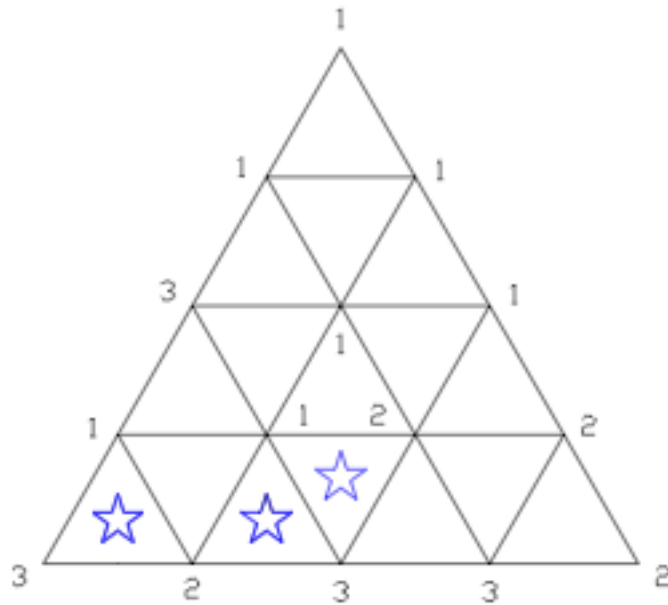
each inside door is counted twice. On the other hand, the doors to the outside of the house are only counted once. Therefore, if  $I$  is the number of inside doors and  $O$  is the number of outside doors, then  $d_1 + d_2 + \dots + d_k = 2I + O$ . Since  $d_1 + d_2 + \dots + d_k$  is even and  $2I$  is certainly even, we see that  $O = (d_1 + d_2 + \dots + d_k) - 2I$  must be even. This is exactly what we wanted to prove!

The following invariant turns out to be useful in the field of topology:

**Problem 12** Consider a triangle which is subdivided into smaller triangles, with each vertex of each triangle assigned a number 1, 2 or 3 in such a way that:

- (a) each vertex of the big triangle receives a different number, and
- (b) on the edge of the big triangle whose endpoints are labeled  $a$  and  $b$ , all other vertices lying on that edge are labeled either  $a$  or  $b$ .

Prove that one of the smaller triangles must have vertices labeled 1, 2, and 3:



Get started by proving that on the 1-2 side of the big triangle, there are an odd number of small edges labeled (1, 2) (this is the one-dimensional version of the theorem you are proving). Now use problem 11 where the “house” is the big triangle, the “rooms” are the small triangles, and the “doors” are the edges labeled (1, 2).

*Solution.* We'll follow the hint, which says we should prove that there are an odd number of small edges labeled  $(1, 2)$  on the 1-2 side of the big triangle. Let  $a_1, a_2, \dots, a_n$  be the numbers on the vertices which lie on the 1-2 side of the big triangle, in order (In particular,  $a_1 = 1, a_n = 2$ ). Note that  $a_n - a_1 = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3}) + \dots + (a_2 - a_1)$  because of all of the cancellations. The sum of the  $n - 1$  terms in parentheses shown above is an odd number, and this means that an odd number of the terms in parentheses must be odd. But since each  $a_i$  is either 1 or 2, the terms in parentheses can be either  $-1, 0$ , or  $1$ . So, an odd number of the terms in parentheses must be either  $1$  or  $-1$ . But these terms correspond exactly to the small edges labeled  $(1, 2)$ .

Now we follow the next hint, and call all the small triangles rooms, and all the edges of them which are labeled  $(1, 2)$  doors. By the first paragraph, there are an odd number of doors leading outside of the house (none of the small edges on the other sides of the big triangle are labeled  $(1, 2)$  by the problem statement). Therefore, problem 11 shows that one of the rooms on the inside of the house must have an odd number of doors. It's certainly impossible for a small triangle to have three doors, so this means that one of the small triangles has exactly one door. The only way this is possible is if its vertices are labeled 1, 2, and 3.

This result is called Sperner's lemma, and was used by Emmanuel Sperner to simplify the proof of a result called Brauer's Fixed Point Theorem.