

- (3) We will now prove the inequality between the arithmetic and the harmonic mean:

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}$$

- (a) Multiply both sides of the inequality by 2:

$$2 \times \frac{a+b}{2} \geq 2 \times \frac{2ab}{a+b}$$
$$a+b \geq \frac{4ab}{a+b}$$

- (b) Multiply both sides of the inequality above by  $a+b$ .

$$(a+b)(a+b) \geq (a+b) \frac{4ab}{a+b}$$
$$(a+b)^2 \geq 4ab$$

- (c) Use the formulas from the warm up to expand the left-hand side of the inequality:

$$a^2 + 2ab + b^2 \geq 4ab$$

- (d) Subtract  $4ab$  from both sides of the inequality:

$$a^2 + 2ab + b^2 - 4ab \geq 4ab - 4ab$$
$$a^2 - 2ab + b^2 \geq 0$$

- (e) Use the formulas from the warm up to simplify your answer:

$$(a-b)^2 \geq 0$$

- (f) Explain why the inequality in (e) is always true.

Any number squared is greater than or equal to zero so  $(a-b)^2 \geq 0$ .

- (4) Suppose Bronwen and Saida are mowing lawns. They can't decide whether it is more fair for them to each mow the same area of lawns or if it is more fair for them to mow lawns for the same amount of time. As a result, they decide to choose the method that would allow them to have the greatest rate while still being fair. Suppose Bronwen can mow 2 lawns an hour and Saida can mow 1 lawn an hour.

- (a) Based on the proof you did above, which method do you think will be faster? Why?

Arithmetic means  $\geq$  harmonic means. Since we want the fastest rate, we want the arithmetic mean of the rates. Arithmetic means of the rates correspond to same amount of time of work.

- (b) Suppose they decide to each mow the same area of a lawn.

- (i) How long would it take Bronwen to mow her half of the lawn?

Bronwen can mow 2 lawns in 1 hour so it would take her  $\frac{1}{4}$  hour to mow  $\frac{1}{2}$  lawn.  
so  $\frac{1}{4}$  hour = 15 minutes.

- (ii) How long would it take Saida to mow her half of the lawn?

Saida can mow 1 lawn in 1 hour so it would take her  $\frac{1}{2}$  hour to mow  $\frac{1}{2}$  lawn.  
so  $\frac{1}{2}$  hour = 30 minutes.

- (iii) How long would it take in total for them to mow their halves of the lawn?

15 minutes + 30 minutes = 45 minutes.

- (iv) What is the average rate at which they can mow lawns if they each mow half a lawn?

45 minutes / lawn =  $\frac{3}{4}$  hour / lawn  
so  $\frac{4}{3}$  lawns / hour.

- (v) Is this average the arithmetic mean or the harmonic mean?

Harmonic mean

(c) Suppose they decide to each mow a lawn for the same amount of time.

(i) How many lawns can Bronwen and Saida mow together in one hour?

Together, they can mow  $2+1$  lawns in one hour. So 3 lawns / hour.

(ii) How long would it take in total for them to mow a lawn?

$\frac{1 \text{ hour}}{3 \text{ lawns}} = \frac{1}{3} \text{ hour / lawn} = 20 \text{ minutes / lawn}$

(iii) What is the average rate at which they mow lawns if they work the same amount of time?

2 lawn in 1 hour  
+ 1 lawn in 1 hour  

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3 lawn in 2 hours

So 1.5 lawns/hour

(iv) Is this average the arithmetic mean or the harmonic mean?

arithmetic mean.

(d) Which rate is faster?

Arithmetic mean is faster.

(5) **Challenge Problem:** Yash and Austin are typesetting a book. Yash works with the speed of  $p_1$  pages per hour. Austin works with the speed of  $p_2$  pages per hour. Yash typesets the first half of the book. Austin typesets the second half of the book. How many pages per hour does one have to typeset in order to typeset a book of the same length in the same amount of time? (Hint: Try using  $n$  to represent the number of pages in the book.)

(a) What portion of the book can Yash typeset in 1 hour?

$$\frac{p_1}{n}$$

(b) What portion of the book can Austin typeset in 1 hour?

$$\frac{p_2}{n}$$

- (c) How long will it take Yash to typeset half of the book?

$$\frac{n}{2p_1}$$

- (d) How long will it take Austin to typeset half of the book?

$$\frac{n}{2p_2}$$

- (e) What is the total amount of time Yash and Austin will spend typesetting the book?

$$\text{time} = \frac{n}{2p_1} + \frac{n}{2p_2} = \frac{n}{2} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)$$

- (f) Find the rate of typesetting at which one needs to work in order to typeset the same book in the same amount of time.

$$\frac{n}{r} = \frac{n}{2} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)$$
$$r = \frac{2}{\frac{1}{p_1} + \frac{1}{p_2}}$$