NONCLASSICAL CONSTRUCTIONS II

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Now we will try ourselves on Poncelet-Steiner constructions. You can only use an (unmarked) straight-edge but you can assume that somewhere in the plane there is one circle (and its center) given.

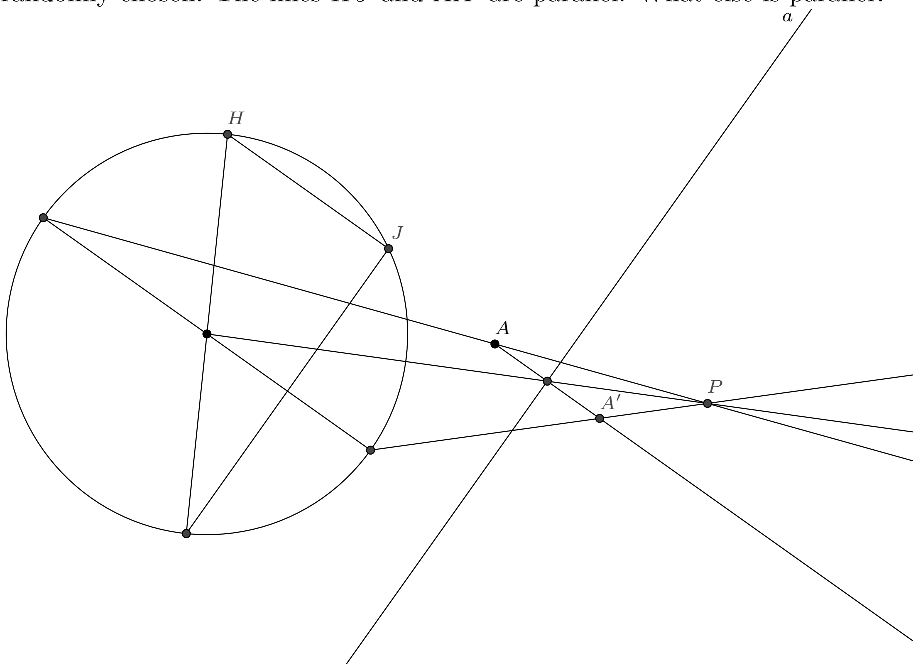
Problem 1 (Construction of parallels through a given point)**.** (i) Just using an (unmarked) straight-edge construct the parallel to a bisected line segment (that is a line segment of which you are given the midpoint).

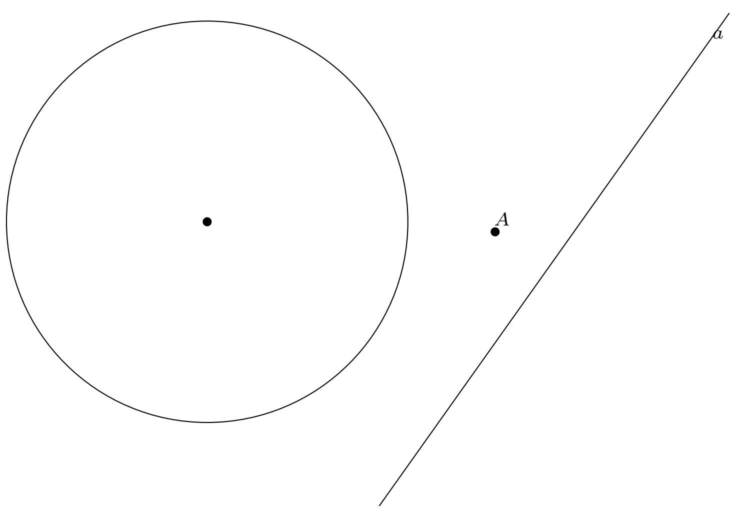
(ii) Now assume you are given a circle and its midpoint. Construct a parallelogram inscribed in the circle. *Hint:* Start with a random diameter of the circle and try using the previous result.

(iii) You are still given a circle and its midpoint. Construct a bisected line segment on a given line (not necessarily passing through the circle). *Hint:* Choose a random diameter and intersect it with the line. Now try using what you've learned so far.

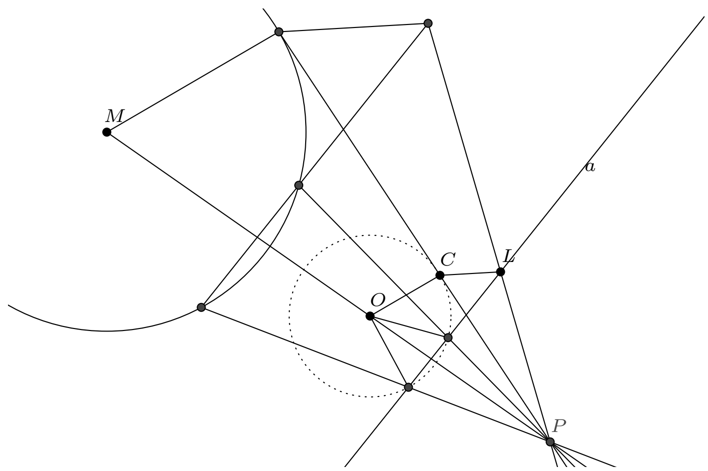
(iv) How does this enable you to give the Poncelet-Steiner construction of the parallel to a given line through a given point?

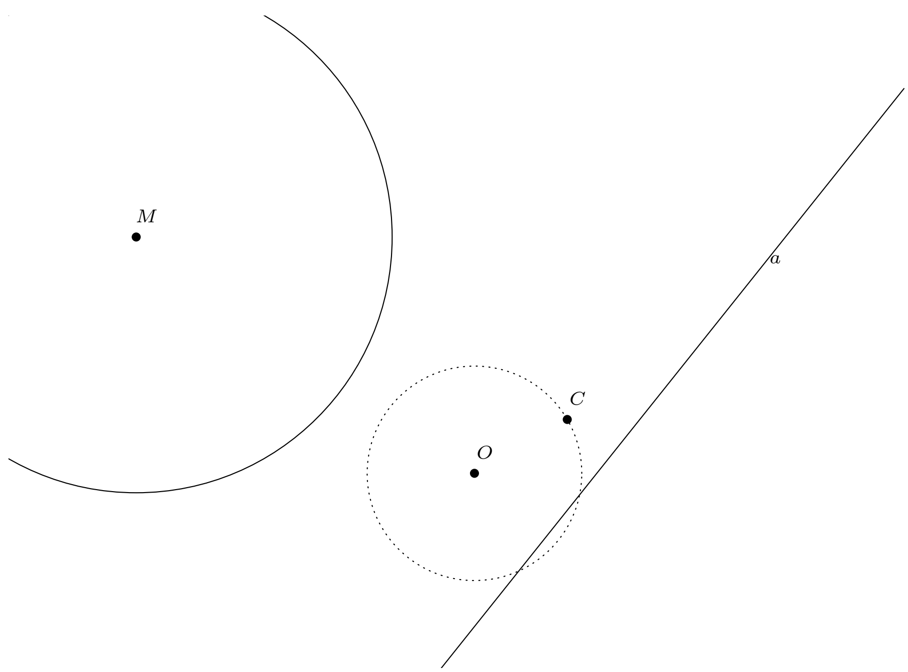
Problem 2 (Reflection on a line)**.** Given a point A and a line a, use the sketch to construct its reflection A'. The second sketch shows you what you start out with. *Hint:* The diameter containing H is randomly chosen. The lines HJ and AA' are parallel. What else is parallel?





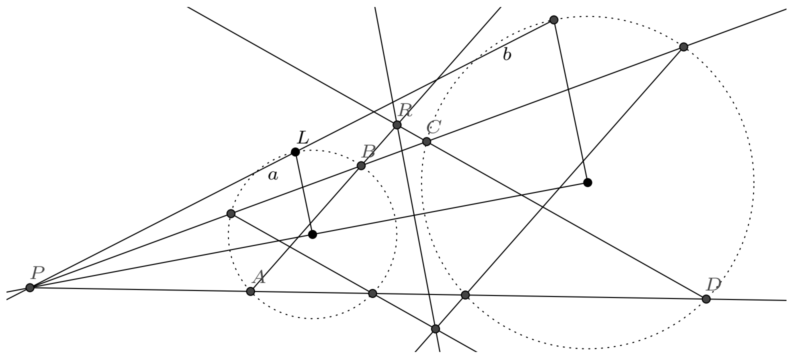
Problem 3 (Intersection of a line and a circle)**.** Use the following sketch to construct the intersection points of the circle around O of radius $\overline{OC}$ and the line a. The second sketch shows what you start out with. The dotted circle cannot be drawn (with a straight-edge), so it's imaginatively there but you cannot intersect lines with it directly. Include a proof why your construction works. *Hint:* The point L is freely chosen anywhere on a. Things that look parallel, are most likely parallel. First use any apparent parellelities and the intersect theorem to prove your assertion. Then use the intersect theorem (several times) to prove that all the lines you need are parallel.

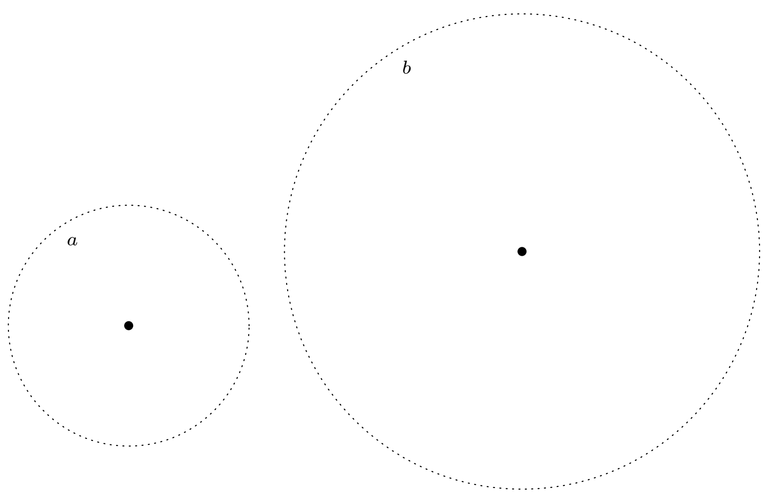




Problem 4. Refer to the appendix for the definition and basic properties of the radical axis of two circles. Show that the (three different) radical axis of three circles intersect in one point. Can you use this to construct the radical axis of two circles using straight edge and compass?

Problem 5 (Radical axis of two circles)**.** Given two circles a and b, use the sketch below to construct their radical axis. The second sketch shows what you start out with. The dotted circles are given (but not drawn). However, thanks to the previous problem you can merrily intersect any line with them. *Hint:* The point L is chosen freely on a. So are the rays containing A and B. Show that $ABCD$ is concyclic and conclude from the previous problem that therefore R must be on the radical axis. Can you identify a second point on the radical axis? During your construction, be mindful that the only thing you know about the given circles is their midpoint and radius. In particular you can NOT draw them/intersect them with other circles.





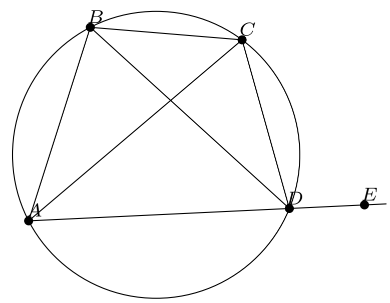
Problem 6. Use the previous two constructions to construct the intersection points of two circles and thus finish the proof of the Poncelet-Steiner Theorem.

Appendix: Some Facts from Geometry

Recall the following facts from geometry. If you're curious you can ask one of the instructors to give you proofs for them.

Concyclic Points

In the sketch below the points A, B, C, and D are on a circle if and only if the angles $\sphericalangle ABD$ and $\sphericalangle ACD$ agree. Similarly A, B, C, and D are on a circle if and only if the angles $\sphericalangle ABC$ and $\sphericalangle EDC$ agree.



Radical Axis

The radical axis of two circles c_1 and c_2 is the set of points P such that the tangents through P to c_1 and c_2 have the same length. Note that we can draw two different tangents through P to a given circle, but they will always have the same length. It can be shown that the radical axis is a line perpendicular to the line connecting the centers of c_1 and c_2. Can you describe the radical axis of two intersecting circles?

The sketch below shows the radical axis of two circles and a point P on it such that the tangential lengths r_1 and r_2 agree.