

Invariants II

LA Math Circle (Advanced)

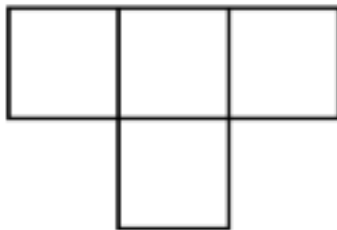
November 15, 2015

Recall that an invariant is some quantity associated with a system that is left unchanged by a specified process. We typically use them to show that no matter how long this process goes on, the system will never be in a certain state. Let's solve some problems!

Problem 1 On a tropical island there are 20 red chameleons, 13 blue chameleons, and 10 green chameleons. When two chameleons of different color meet, they change to the third color. Can all the chameleons eventually be of the same color?

Problem 2 Consider an 8×8 chessboard with two opposite corners removed. Can you cover the chessboard with dominoes? (Hint: First color the chessboard like in real life. Find an invariant related to the colors of the squares.)

Problem 3 Prove that a 10×10 board cannot be covered by T-shaped tetrominos (see the picture).



Problem 4 Let n be an odd natural number. Start with the numbers $1, 2, \dots, 2n$ and perform the following operation: cross off any two of them, say they are a and b , and replace them with the number $|a - b|$. Do this until only one number remains. What is the parity of the last number?

Problem 5 An $n \times m$ table is filled with real numbers such that each row and column add up to 1. Prove that $m = n$.

Problem 6 Begin with the number

9, 999, 333, 666, 369, 999, 999, 999, 999, 991, 999, 999, 333, 369.

If you add up all of its digits, then add up all the digits of the resulting number, and so on, will you ever get the number 12? (No credit for actually doing the computation, you must find an invariant! Hint: divisibility criterion.)

Problem 7 A delicious cherry pie is divided into 6 equal slices. One fresh cherry is placed on top of each slice. Jeff wants to play with the cherries on top to try to put them all on his slice. His mom, a clever mathematician, will only allow this under the condition that Jeff can only shift any two cherries to slices bordering those they stand on at the moment (he must move them at the same time). He can do this as many times as he likes. Can he meet his goal? (Hint: start by numbering the slices 1 through 6. Try to find an invariant based on the numbers of the slices containing cherries.)

Problem 8 Can a 10×10 chessboard be covered up with 4×1 tiles? (Hint: to find an invariant, start by writing the numbers $-1, 3, -1, -1, -1, 3, -1, -1, -1, 3$ in the first row of the chessboard; then the same sequence in the second row, but shifted one spot to the left so that it starts with 3, then the same sequence in the third row shifted one more spot to the left, etc.)

Problem 9 Begin with the numbers 1, 2, and 4. Now take any two of the numbers a, b , cross them out, and replace them with the numbers $.6a + .8b$ and $.8a - .6b$. Repeat this operation as many times as you like. Can you ever obtain the numbers 2, 3, 3?

Invariants can also help us to solve physical problems. Consider an object with mass m , moving at speed v and whose vertical distance from the ground is h . Let $g = 9.8m/s^2$ be the acceleration of gravity. Then experiments have shown that the quantity

$$\frac{1}{2}mv^2 + mgh$$

does not change unless the object interacts with another object. The quantity $\frac{1}{2}mv^2$ is called the object's *kinetic energy*, the quantity mgh is called the object's *potential energy*, and their sum is simply called the object's *energy*. Consequently, this law is called the Law of Conservation of Energy.

Problem 10 Consider a roller coaster which starts from rest at 300 meters in the air, and which does not use a motor since it is unaffected by friction. The roller coaster then descends 100 meters, goes through three loopdy-loops, then a hairpin turn, then goes up a minor hill. At the top of this hill, its height is 140 meters. Use the law of conservation of energy to determine the speed of the coaster at the top of the hill.

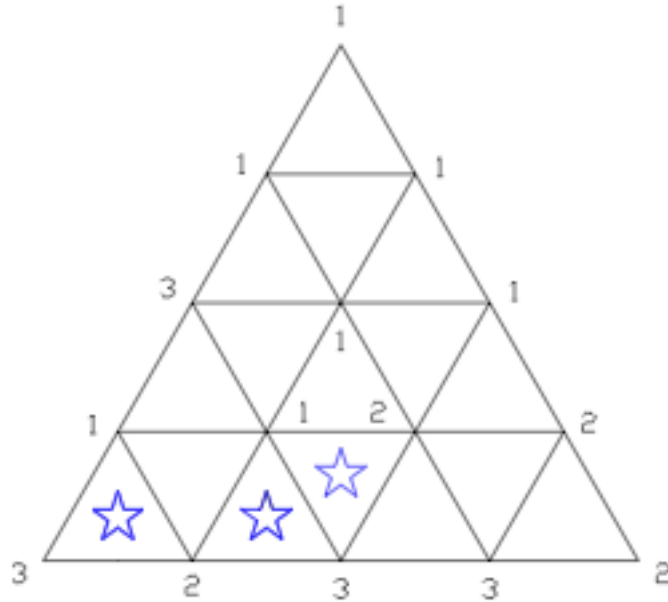
Problem 11 Prove that if every room in a house has an even number of doors, then there must be an even number of doors on the outside of the house.

The following invariant turns out to be useful in the field of topology:

Problem 12 Consider a triangle which is subdivided into smaller triangles, with each vertex of each triangle assigned a number 1, 2 or 3 in such a way that:

- (a) each vertex of the big triangle receives a different number, and
- (b) on the edge of the big triangle whose endpoints are labeled a and b , all other vertices lying on that edge are labeled either a or b .

Prove that one of the smaller triangles must have vertices labeled 1, 2, and 3:



Get started by proving that on the 1-2 side of the big triangle, there are an odd number of small edges labeled (1, 2) (this is the one-dimensional version of the theorem you are proving). Now use problem 11 where the “house” is the big triangle, the “rooms” are the small triangles, and the “doors” are the edges labeled (1, 2).