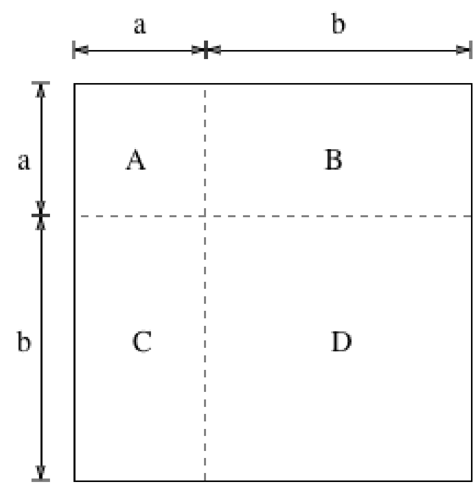
ARITHMETIC AND HARMONIC MEANS

BEGINNERS NOVEMBER 15, 2015

Warm Up Problems

(1) Look at the picture below.



(a) Given that the edge of a square is $a+b$, what is the area of the square?

$Area = (a+b)^2$

(b) We now want to find the areas of 4 smaller rectangles that make up the square with side length $a+b$.

(i) What is the area of A?

$Area_A = a^2$

(ii) What is the area of B?

$Area_B = ab$

(iii) What is the area of C?

$Area_C = ab$

(iv) What is the area of D?

$Area_D = b^2$

(c) Find the total area of the square of side length $a+b$ using the answers you got in part (b). Remember to simplify the formula!

$Area = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

(d) Using the answers you got in parts (a) and part (c), write down a formula for $(a+b)^2$:

$(a+b)^2 = a^2 + 2ab + b^2$

(e) To test the formula, pick values for a and b:

$a = 2$ (Answers may vary)

$b = 3$ (Answers may vary)

(f) Evaluate the left-hand side of the formula:

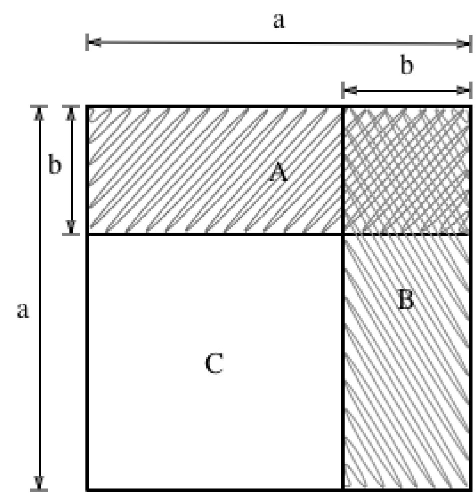
$(a+b)^2 = (2+3)^2 = 5^2 = 25$ (Answers may vary)

(g) Evaluate the right-hand side of the formula:

$a^2 + 2ab + b^2 = 2^2 + 2 \cdot 2 \cdot 3 + 3^2 = 4 + 12 + 9 = 25$ (Answers may vary)

(h) Are the two values you got equal to each other?
Yes!

(2) Consider the square made up of overlapping shapes below. (On the picture, A and B are both rectangles with sides a and b. They overlap over a square with side length b).



(a) Given that the edge of a square is a, what is the area of the square?

$Area = a^2$

(b) We now want to find the areas of the rectangles that make up the square with edge of length a.

(i) What is the area of A?

$Area_A = ab$

(ii) What is the area of B?

$Area_B = ab$

(iii) What is the area of C?

$Area_C = (a - b)^2$

(iv) What is the area of of the region where A overlaps with B?

$Area_{overlap} = b^2$

(c) Use answers from part (b) to evaluate the total area of the square with edge of length a. Remember to simplify the formula!
$Area = ab + ab + (a-b)^2 - b^2 = 2ab + (a-b)^2 - b^2$
The reason why we subtract b^2 is because we double count the overlapped area when we add the areas of A and B so we need to subtract the double counted area once.

(d) Using the answers you got in part (a) and part (c), write an formula relating the two areas:

$a^2 = 2ab + (a-b)^2 - b^2$

(e) Now rewrite the formula you got in part (d) so that $(a-b)^2$ is on the left-hand side and everything else is on the right-hand side:

$(a-b)^2 = a^2 - 2ab + b^2$

(f) To test the formula, pick values for a and b:

$a = 8$ (Answers may vary)

$b = 5$ (Answers may vary)

(i) Evaluate the left-hand side of the formula in (e):

$(a-b)^2 = (8-5)^2 = 3^2 = 9$

(Answers may vary)

(ii) Evalue the right-hand side of the formula in (e):

$a^2 - 2ab + b^2 = 8^2 - 2 \cdot 8 \cdot 5 + 5^2 = 64 - 80 + 25 = 9$

(Answers may vary)

(iii) Do the two values you got equal to each other?

Yes!

Harmonic Means

(1) The speed of a motor boat moving downstream is v_1. The speed of the same boat moving upstream is v_2.

(a) Find the speed of the boat in still water:

$v = \frac{v_1+v_2}{2}$

(b) Suppose the boat moves for time t_1 downstream and then moves upstream for time t_2. Find the average speed of the boat during this trip:

(i) Find the distance the boat travels downstream:

$d_1 = v_1 t_1$

(ii) Find the distance the boat travels upstream:

$d_2 = v_1 t_2$

(iii) Find the total distance the boat travels:

$d_{total} = v_1 t_1 + v_2 t_2$

(iv) Find the total time the boat travels:

$t_{total} = t_1 + t_2$

(v) Find the average speed by dividing the total distance by the total time of travel:

$u = \frac{v_1 t_1 + v_2 t_2}{t_1 + t_2}$

(vi) Is it true that u equals to the speed of the boat in still water that you found above in part (a)? Why or why not?

No. One way to see this is if t_1 is really small and t_2 is really large.

Then u would be much closer to v_2 than the average of v_1 and v_2.

(vii) What should be true about t_1 and t_2 so that $u = v$?

$t_1 = t_2$

(2) Suppose the distances traveled by the boat upstream and downstream are equal.

(a) Relate the distance traveled by the boat upstream and downstream using v_1, t_1, v_2 and t_2.

If the distances are equal then $d_1 = d_2$. Replacing d_1 and d_2 with the values from part (b) gives:

$v_1 t_1 = v_2 t_2$

(b) Using the equality you got in part (a), write t_2 in terms of v_1, t_1 and v_2:

$t_2 = \frac{v_1 t_1}{v_2}$

(c) Write the total time of travel using t_1 and t_2. Then rewrite it in terms of t_1, v_1 and v_2 using the equality you got from part (b):

$t = t_1 + t_2 = t_1 + \frac{v_1 t_1}{v_2}$

(d) Use the fact that distances traveled upstream and downstream are equal to write the expression for the total distance it in terms of v_1 and t_1:

$d = v_1 t_1 + v_2 t_2 = v_1 t_1 + v_1 t_1 = 2 v_1 t_1$ (because $v_2 t_2 = v_1 t_1$)

(e) Now find the average speed by dividing the total distance by the total time of travel.

$speed = \frac{distance}{time} = \frac{d}{t} = \frac{2 v_1 t_1}{t_1 + \frac{v_1 t_1}{v_2}}$

(f) Now simplify the answer you obtained in part (e). This will be shown on the board.

$speed = \frac{2v_1t_1}{t_1+\frac{v_1t_1}{v_2}}$

$= \frac{2v_1}{1+\frac{v_1}{v_2}}$ (by cancelling out t_1)

$= \frac{2}{\frac{1}{v_1}+\frac{1}{v_2}}$ (by dividing v_1 from the numerator and denominator)

This expression is called the **harmonic mean** (or **harmonic average** of two numbers). Harmonic mean h of a and b has the property that its inverse is the (arithmetic) average of the inverses of a and b:

$$\frac{1}{h} = \frac{\frac{1}{a}+\frac{1}{b}}{2}$$

Harmonic mean comes up very often when you solving problems related to speed and rate of work.

(1) Use the formula $h = \frac{2ab}{a+b}$ to show that $\frac{1}{h} = \frac{\frac{1}{a}+\frac{1}{b}}{2}$.

$\frac{1}{h} = \frac{a+b}{2ab}$ (by inverting the first equation)

$= \frac{1+\frac{b}{a}}{2b}$ (by dividing a from the numerator and denominator)

$= \frac{\frac{1}{b}+\frac{1}{a}}{2}$ (by dividing b from the numerator and denominator)

$= \frac{\frac{1}{a}+\frac{1}{b}}{2}$ (by rewriting the equation)

(2) Mary can decorate 6 cakes an hour. Joy can decorate 12 cakes an hour. For the Thanksgiving dinner, Mary started decorating the cakes and did half of the job. After that, Joy finished the other half.
  (a) Suppose they had to decorate 24 cakes. How long would it take Mary to decorate half of the 24 cakes? How long would it take Joy to decorate half of the 24 cakes?
  Half of 24 cakes is 12 cakes.
  Since Mary can decorate 6 cakes in an hour, we can find the time it takes Mary to decorate 12 cakes by doing $\frac{12 \text{ cakes}}{\frac{6 \text{ cakes}}{1 \text{ hour}}} = 2$ hours.
  Similarly, It would take Joy 1 hour to decorate 12 cakes.

(b) How long did it take the both of them to decorate 24 cakes?
It took Mary 2 hours and Joy 1 hour so in total, it took them 3 hours to decorate 24 cakes.

(c) How many cakes can they decorate in one hour if each of them decorates half the cakes?
Since they can decorate 24 cakes in 3 hours, then $\frac{24 \text{ cakes}}{3 \text{ hours}} = \frac{8 \text{ cakes}}{1 \text{ hour}}$. In other words, 8 cakes per hour.

(d) Does the answer you got in part (c) change if they had to decorate 48 cakes? How about n cakes?
Half of 48 cakes is 24 cakes.
It would take Mary 4 hours to decorate 24 cakes and Joy 2 hours to decorate 24 cakes. In total, it would take them 6 hours to decorate 48 cakes. So they can decorate 8 cakes per hour.
Similarly, half of n cakes is $\frac{n}{2}$ cakes.
It would take Mary $\frac{\frac{n}{2}}{6} = \frac{1}{2} \cdot \frac{n}{6}$ hours to decorate $\frac{n}{2}$ cakes and Joy $\frac{\frac{n}{2}}{12} = \frac{1}{2} \cdot \frac{n}{12}$ hours to decorate $\frac{n}{2}$ cakes. In total, it would take them $\frac{1}{2} \cdot \frac{n}{6} + \frac{1}{2} \cdot \frac{n}{12} = \frac{n}{2} \cdot (\frac{1}{6} + \frac{1}{12}) = \frac{n}{2} \cdot \frac{1}{4} = \frac{n}{8}$ hours to decorate n cakes. Cancelling out n, we get it would take $\frac{1}{8}$ hours to decorate 1 cake. Or in other words, 8 cakes per hour.
As can be seen, the rate does not change.

(e) Kara was working for the same amout of time as Mary and Joy combined and decorated the same total number of cakes. How many cakes an hour can Kara decorate?
8 cakes per hour.

(3) It takes Ishita 3 hours to bike an entire bike path. It takes Luke $4\frac{1}{2}$ hours to bike the same path. Suppose they start from the opposite ends of the bike path at the same time. How soon will they meet?

(a) What portion of the path's length does Ishita cover in 1 hour?
$\frac{1}{3}$ of the path per hour

(b) What portion of the path's length does Luke cover in 1 hour?
$\frac{1}{4.5} = \frac{2}{9}$ of the path per hour

(c) What portion of the path do Luke and Ishita cover together in 1 hour?
$\frac{1}{3} + \frac{2}{9} = \frac{5}{9}$ of the path per hour

(d) How soon will they meet if they start on the opposite sides of the path at the same time? Express your answer in hours and minutes.
$\frac{9}{5}$ hours per path which is 1 hour and 48 minutes.

(4) It takes Aiden a hours to do the job. It takes Max b hours to do the same job.

(a) How many hours will it take them to finish the work together. (*Tip:* think about what portion of the job each of them will do):

(i) Aiden does $\frac{1}{a}$ portion of the job in 1 hour;

(ii) Max does $\frac{1}{b}$ portion of the job in 1 hour;

(iii) Together, they do $\frac{1}{a} + \frac{1}{b}$ portion of the job in 1 hour. Simplify the expression by using common denominator:

Common denominator is ab so $\frac{1}{a} + \frac{1}{b} = \frac{b}{ab} + \frac{a}{ab} = \frac{b+a}{ab} = \frac{a+b}{ab}$

(iv) It will take them $\frac{1}{\frac{1}{a}+\frac{1}{n}} = \frac{ab}{a+b}$ hours to finish the job.

(b) Suppose Mark and Lev work at the same rate and can finish the same job in the same amount of time that it takes Aiden and Max working together. How many hours will it take for each of them to finish the job alone?
Since two people working at the same rate have rate $\frac{a+b}{ab}$, then one person working would be half that rate, which is $\frac{a+b}{2ab}$. Notice how this is similar to the equation for $\frac{1}{h}$!

Comparing Arithmetic and Harmonic Means.

(1) Calculate the arithmetic mean and the harmonic means

$$m = \frac{a+b}{2}, \qquad h = \frac{2ab}{a+b}$$

of the numbers below and state which is larger.

(a) $2, 8$

$m = 5$

$h = 3.2$

(b) $3, 1$

$m = 2$

$h = 1.5$

(c) $8, 12$

$m = 10$

$h = 9.6$

(d) $5, 5$

$m = 5$

$h = 5$

(2) Suppose that a and b are both positive. Is there an inequality relating m and h that is true in all of the examples above?

$m \geq h$