

Invariants I

LA Math Circle (Advanced)

November 6, 2015

Let's take a minute to consider the following problem:

Problem 0 There are currently 84 people in the UCLA Mathematical Sciences building. Each minute, either 6 people enter the building or 12 people leave it. Is it possible that some time in the next hour, there will be exactly 100 people in the building?

A good place to start on this problem (and in fact most problems) is to experiment: Try to find a way of having exactly 100 people in the building. Do you think it's possible? If so, show how it can be done! If not, try to figure out why.

Is there anything special about the specific numbers used in the problem? Do they have any special properties that might make having 100 people in the building possible or impossible?

An *invariant* is a quantity associated with a collection of objects which does not change when we perform a certain type of action on those objects. Invariants are useful because they can tell us that it is impossible for something to happen - like there being 100 people in a building.

For instance, in problem 0, the collection of objects was the collection of all people inside the math department. Two actions were performed on that collection: either 6 people were added, or 12 people were removed. These actions *did not change* whether the number of people in the building was divisible by three.

Let's try one more problem as a class. You should try experimenting with smaller numbers first!

Problem 1 There are Martian amoebae of three types in a test tube: A, B, and C. Two amoebae of any two different types can merge into one amoeba of the third type. After several such merges, only one amoeba remains in the test tube. There were initially 20 amoebae of type A, 21 of type B, and 22 of type C. What is the type of the final amoeba?

Problem 2 Let's play a game. Consider the following list of numbers (there are six 0's and five 1's):

0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1

(a) Perform the following operation ten times: cross out any two numbers, and

- If the two numbers were equal, add a 0 to the end of the list.
- If the two numbers were not equal, add a 1 to the end of the list.

SHOW YOUR WORK. What number are you left with?

(b) Perform the same operation ten times, but in a different order than you did in (a).

(c) Prove that you will always get the same result (Hint: look at (a) and (b) and consider what happens to the sum of all the numbers each time you perform the operation).

Problem 3 (a) Begin with a 4×4 grid of squares. Color the following squares black: Row 1: Column 2; Row 2: Columns 1, 3, 4; Row 3: Column 2; Row 4: Column 2. Now try to make the board entirely the same color by changing the colors of all squares in a row or changing the colors of all the squares in a column as many times as you need. Is it possible? If so, show how, if not, prove it!

(b) Solve the same problem, but this time start with only the top left corner shaded.

Problem 4 Begin with the numbers 1, 2, and 4. Now take any two of the numbers a, b , cross them out, and replace them with the numbers $.6a + .8b$ and $.6a - .8b$. Repeat this operation as many times as you like. Can you ever obtain the numbers 2, 3, 3?

Problem 5 Consider an 8×8 chessboard with two opposite corners removed. Can you cover the chessboard with dominoes (e.g. rectangles which are two squares long and one square wide)?

Problem 6 On a tropical island there are 20 red chameleons, 13 blue chameleons, and 10 green chameleons. When two chameleons of different color meet, they change to the third color. Can all the chameleons eventually be of the same color? (Hint: Think about the number of red minus the number of blue chameleons.)

Problem 7 Let a and b be integers. Define mathematically the phrase “ a divides b ” (Note: b is divisible by a means the same thing).

Problem 8 (a) List the first 5 positive integers divisible by 3.

(b) List the first 4 negative integers divisible by 7.

Let a, b , and n be integers. We say that a is congruent to b modulo n , written $a \equiv b \pmod{n}$, if n divides $b - a$.

Problem 9 (a) List 5 numbers which are congruent to 3 modulo 8.

(b) List 4 negative numbers which are congruent to 2 modulo 5.

(c) For $a = 12, 47$, and $1,000$, find the smallest non-negative number b such that $a \equiv b \pmod{6}$.

Problem 10 In each case, find the smallest non-negative number which is congruent to the sum modulo the given number:

(a) $4 + 3 \equiv \quad \pmod{6}$, $4 + 9 \equiv \quad \pmod{6}$, $-2 + 21 \equiv \quad \pmod{6}$

(b) $12 + 3 \equiv \quad \pmod{5}$, $7 + 18 \equiv \quad \pmod{5}$, $-3 + 8 \equiv \quad \pmod{5}$

(c) $4 + 13 \equiv \quad \pmod{9}$, $13 + 4 \equiv \quad \pmod{9}$, $-5 + 4 \equiv \quad \pmod{9}$

(d) What was the point of these exercises?

Problem 11 Prove that if $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, then $a + b \equiv a' + b' \pmod{n}$. After you prove it, write a sentence explaining why this is important.

Problem 12 Now prove by induction that if $a_i \equiv a'_i \pmod{n}$ for $i = 1, 2, \dots, n$, then $a_1 + a_2 + \dots + a_n \equiv a'_1 + a'_2 + \dots + a'_n \pmod{n}$.

Problem 13 Begin with the number

9, 999, 333, 666, 369, 999, 999, 999, 999, 991, 999, 999, 333, 369.

If you add up all of its digits, then add up all the digits of the resulting number, and so on, will you ever get the number 13? (Hint: please do not just add up all the digits! Instead, investigate the difference $n - s(n)$ between a number n and the sum of its digits $s(n)$ modulo a certain number.)

Invariants are also extremely important in physics. Consider an object with mass m , moving at speed v and whose vertical distance from the ground is h . Let $g = 9.8m/s^2$ be the acceleration of gravity. Then experiments have shown that the quantity

$$\frac{1}{2}mv^2 + mgh$$

does not change unless the object interacts with another object. The quantity $\frac{1}{2}mv^2$ is called the object's *kinetic energy*, the quantity mgh is called the object's *potential energy*, and their sum is simply called the object's *energy*. Consequently, this law is called the Law of Conservation of Energy.

Problem 14 Consider a roller coaster which starts from rest at 300 meters in the air, and which does not use a motor since it is unaffected by friction. The roller coaster then descends 100 meters, goes through three loopdy-loops, then a hairpin turn, then goes up a minor hill. At the top of this hill, its height is 140 meters. Use the law of conservation of energy to determine the speed of the coaster at the top of the hill.