ARITHMETIC MEAN

BEGINNERS NOVEMBER 8, 2015

What is arithmetic mean?

(1) What is the average of the following two numbers:
- 3 and 5

  $\frac{3+5}{2} = \boxed{4}$
- 8 and 12

  $\frac{8+12}{2} = \boxed{10}$
- $\frac{1}{3}$ and $\frac{2}{3}$

  $\frac{\frac{1}{3}+\frac{2}{3}}{2} = \boxed{0.5}$
- 2.5 and 3

  $\frac{2.5+3}{2} = \boxed{2.75}$
- -1 and 1

  $\frac{-1+1}{2} = \boxed{0}$
- -8 and -2

  $\frac{-8+(-2)}{2} = \boxed{-5}$
- -3 and 11

  $\frac{-3+11}{2} = \boxed{4}$

(2) A bag containing 10 apples weighs 3 kg. How much does each apple weigh approximately?

$\frac{3}{10} = \boxed{0.3 \text{ kg}}$

(3) You picked 27 apples. Assuming the apples are the same as the ones in the prevoius question, how much do you expect these apples to weigh?

$27 \times 0.3 = \boxed{8.1\text{ kg}}$

(4) Luke and Dani went trick-or-treating last week. Before they left, they made an agreement that they would evenly split all the candy they got.

(a) That night, Luke got 120 pieces of candy but Dani only got 20 pieces of candy. After splitting the candy, how many pieces of candy would they each have?

$\frac{120+20}{2} = \boxed{70\text{ pieces}}$

(b) Suppose Luke got X pieces of candy and Dani got Y pieces of candy. After splitting evenly, how many will each of them have?

$\boxed{\frac{X+Y}{2}} = m$

(5) Suldee and Jason built a ladder by tying two rope ladders together. The ladder they built allowed them to reach the ground from the second floor.

(a) Suppose the first ladder was 2 meters long and the second ladder was 4 meters long.

(i) How tall is each floor?

$\frac{4+2}{2} = \boxed{3\text{ m.}}$ each floor

(ii) If they had to build a ladder that would allow them to reach the ground floor from the fourth floor, how long would it have to be?

$3m \times 4 = 12m$

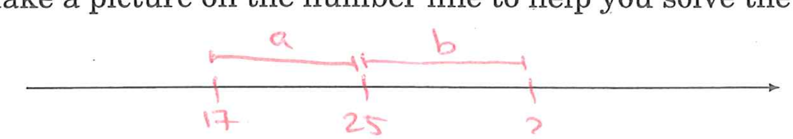
(b) Suppose the first ladder is M meters long and the second is N meters long. How high is one floor?

$\frac{M+N}{2}$

(6) Let a and b be two numbers. Mark these numbers on the number line. Denote the average of these two numbers by m and mark it on the number line as well. Then find the expression for m in terms of a and b.

the segment $\overline{am}$ has the same length as the segment $\overline{mb}$, ~~then~~ the length of $\overline{am} = m-a$, the length of $\overline{mb} = b-m$ so $\overline{am} = \overline{mb} \Rightarrow m-a = b-m \Rightarrow 2m = a+b$
$\Rightarrow m = \frac{a+b}{2}$

(7) The average of two numbers is 25. The smaller number is 17. What is the other number? Make a picture on the number line to help you solve the problem.



$a = 25-7 = 8$.

Since $b=a$, then the value we are trying to find is $25+a = 25+8 = 33$

(8) The average of two numbers is 43. One of the numbers is 67. What is the other number?

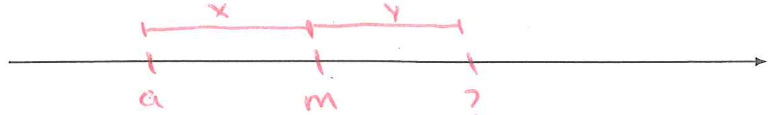
$=b$ $=m$ $=a$

$a - m = 67 - 43 = 24$

$b = m - (a - m) = 43 - 24 = \boxed{19}$

(9) The average of two numbers is m. One of the numbers is a.

(a) Assuming that the average is bigger than this number, i.e., $m > a$, find the other number b. Make a picture on the number line showing a, m and b and use the picture to find b in terms of a and m.

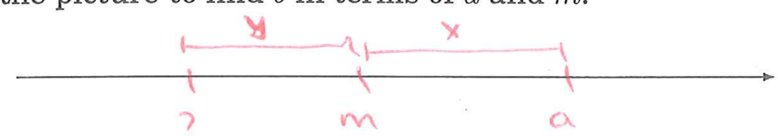


We know that $x=y$ and that $m+y=b$. $y=x=m-a$

so $b=m+y = m+(m-a) = 2m-a$

$\boxed{b=2m-a}$

(b) Assuming that the average is smaller than this number, i.e., $m < a$, find the other number b. Make a picture on the number line showing a, m and b and use the picture to find b in terms of a and m.



We know that $x=y$ and $b=m-y$ so $b=m-x$.

$x=a-m$ so $b=m-(a-m)=2m-a$

$\boxed{b=2m-a}$

Boats On a River

(10) Suppose a boat travels at a speed of 10 m/s in still water. Now we place the boat in a stream. Suppose the speed of stream is of 3 m/s.

(a) What is the speed of the boat when it travels downstream?

$u_{\downarrow} = 10 + 3 = \boxed{13 \text{ m/s}}$

(b) What is the speed of the boat when it travels upstream?

$u_{\uparrow} = 10 - 3 = \boxed{7 \text{ m/s}}$

(11) Suppose a boat travels at the speed of u m/s in still water. Suppose that the speed of current in a stream is v m/s.

(a) Let $u_{\downarrow}$ represent the speed of the boat when travelling downstream. Write $u_{\downarrow}$ in terms of u and v.

$\boxed{u_{\downarrow} = u + v}$

(b) Let $u_{\uparrow}$ represent the speed of the boat when travelling upstream. Write $u_{\uparrow}$ in terms of u and v.

$\boxed{u_{\uparrow} = u - v}$

(12) Suppose we don't know what the speed of a boat is but observe that boats travelling upstream have a speed of $u_\uparrow = 12$ m/s and the boats travelling downstream have a speed of $u_\downarrow = 18$ m/s.

(a) What is the speed of the boat?

The speed of the boat is half way between $u\uparrow$ and $u\downarrow$.

So $u = \frac{12+18}{2} = \boxed{15 \text{ m/s}}$

(b) What is the speed of the current?

$v = u - u\uparrow = 15 - 12 = \boxed{3 \text{ m/s}}$

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$v = u\downarrow - u = 18 - 15 = \boxed{3 \text{ m/s}}$

(13) Suppose we don't know what the speed of a boat, u, is but observe that boats travelling upstream have a speed of $u_\uparrow$ m/s and the boats travelling downstream have a speed of $u_\downarrow$ m/s.

(a) What is the speed of the boat in terms of $u_\uparrow$ and $u_\downarrow$?

$\boxed{u = \frac{u\uparrow + u\downarrow}{2}}$

(b) What is the speed of the current?

$v = u_\downarrow - u = u_\downarrow - \frac{u\uparrow + u\downarrow}{2} = \frac{2u\downarrow}{2} - \frac{u\uparrow + u\downarrow}{2}$

$\boxed{v = \frac{u\downarrow - u\uparrow}{2}}$

Arithmetic Mean With More Than Two Objects

We know from Question 6 that the average of two numbers a and b is a number m such that $a + b = 2m$. In other words,

The sum of two numbers equals to twice their average.

Now we'll look at the averages of more than just two numbers.

(14) Find the average of the following numbers:

(a) $6, 10, 5$

$$\frac{6+10+5}{3} = \frac{21}{3} = \boxed{7}$$

(b) $13, 17, 10, 80$

$$\frac{13+17+10+80}{4} = \frac{120}{4} = \boxed{30}$$

(c) $-10, 10, -20, 20$

$$\frac{-10+10+(-20)+20}{4} = \frac{0}{4} = \boxed{0}$$

(d) a, b, c, d

$$\boxed{\frac{a+b+c+d}{4}}$$

(15) If we had n numbers $a_1, a_2, ..., a_n$, what would the arithmetic mean, m, be in terms of $a_1, a_2, ..., a_n$ and n?

$$m = \frac{\text{sum of all the numbers}}{\text{number of numbers}} = \boxed{\frac{a_1+a_2+\dots+a_n}{n}}$$

(16) Suppose Cory and Emmanuelle each taught a Beginners math circle class and they had a quiz last week. Each of their classes has 20 students. Emmanuelle forgot to tell her class that there would be a quiz so only one student showed up and received 0 points on the quiz. Cory remembered to tell his class that there would be a quiz so as a result 19 students in his class all showed up and everyone received 20 points on the quiz.

(a) What is the average score in Emmanuelle's class?

$\frac{0}{1} = \boxed{0 \text{ points}}$

(b) What is the average score in Cory's class

$\frac{19 \times 20}{19} = \boxed{20 \text{ points}}$

(c) What is the average score obtained by all of the students who took the quiz?

$\frac{20 \times 19 + 0 \times 1}{20} = \boxed{19 \text{ points}}$

Weighted Averages

(17) The average height of girls in math circle is 140 cm. The average height of boys is 150 cm.

(a) Is this information sufficient to find the average height of all the students in math circle? Why or why not?

No, as we don't know the distribution between boys and girls.

(b) Now suppose you know that the number of girls equals to the number of boys. Given this information, can you find the average height of the math circle students?

Yes! If N is the number of girls and the number of boys then

$$m = \frac{N \times 140 + N \times 150}{N \times 2} = \frac{\cancel{N} \times (140+150)}{\cancel{N} \times (2)} = \frac{140+150}{2} = \boxed{145 \text{ cm}}$$

(18) There are 20 students in the Early Elementary group. The average age of students in the Early Elementary group is 7.5 years. There are 40 students in the Junior circle. The average age of students in the Junior circle is 10 years. What is the average age of the students in the two groups combined?

$$\frac{20 \times 7.5 + 40 \times 10}{60} = \frac{550}{60} = 9\tfrac{1}{6} = \boxed{9 \text{ years } 2 \text{months}}$$

(19) Trail mix consists of walnuts and raisins. There are twice as much walnuts as raisins in the mix. The price of walnuts is \$5 per kilogram. The price of raisins is \$3 per kilogram. How should the trail mix be priced?

Want to know price of 1 kg of trail mix.

1 kg raisins : \$3

+ 2 kg walnuts : \$10

3 kg trail mix : \$13

1 kg trail mix = $\frac{\$13}{3}$ = \$4.33

$\boxed{\$4.33 \text{ per kg}}$

(20) Halloween mix consists of candy corn, pretzels and chocolate. In 5 kilograms of Halloween mix, there is 1 kilogram of candy corn, 2 kilograms of pretzels and 2 kilograms of chocolate. The price of candy corn is \$10 per kilogram and the price of pretzels is \$2 per kilogram and the price of chocolate is \$16 per kilogram. How should the Halloween mix be priced?

1 kg candy corn: \$10
2 kg pretzels: \$4
+ 2 kg chocolate: \$32
5 kg halloween mix: \$46
1 kg halloween mix $= \frac{\$46}{5} = \$9.20$

$\boxed{\$9.20 \text{ per kg}}$

(21) Two math circle groups participate in Math Kangaroo. The first group has x students. The average score in this group is A. The second group has y students. The average score in this group is B. What is the average score for all of the students?

Assume all the students in group 1 is score A, group 2 is score B.

Then $m = \frac{\text{sum of all the scores}}{\text{total number of scores}} = \frac{\overbrace{A+A+\dots+A}^{x}+\overbrace{B+B+\dots+B}^{y}}{x+y}$

$= \boxed{\frac{Ax+By}{x+y}}$

(22) **Challenge:** Suppose we have n groups of numbers. The first group of numbers has x_1 numbers and the average of these numbers is a_1. The second group of numbers has x_2 numbers and the average of these numbers is a_2. ... The n^{th} group of numbers has x_n numbers and the average of these numbers is a_n. What is the average of all the numbers in the n groups?

$$\boxed{m = \frac{x_1a_1 + \dots + x_na_n}{x_1 + \dots + x_n}}$$

When Averages Change

(23) The average age of math circle students in Beginners circle is 10 years and $7\frac{1}{2}$ months old. There are 24 students in the class.

(a) What will the average age of the students in 1 year be?

$\boxed{\text{11 years and } 7\frac{1}{2} \text{ months old}}$

(b) Suppose we don't know how many students there are in the class. Can we still figure out what the average age of the students is in 1 year? Why or why not?

As all the students ages increase by 1 year, the average increases by 1 year as well.

(24) Suppose the average height of the students in Ishita's class is 146 cm. In a month, everyone grows by 1 cm. What is the average height of the class in a month?

$146 + 1 = \boxed{147 \text{ cm}}$

(25) The average age of a group of two friends is 30 years old. They then met another friend and now the average age of the group of friends is 40 years old. How old is the new friend?

$$\frac{30+30+X}{3} = 40$$

$$30+30+X = 120$$

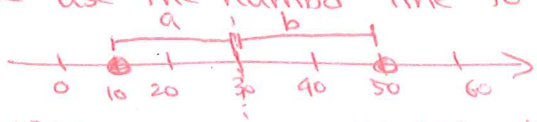
$$X = 60$$

$\boxed{\text{60 years old}}$

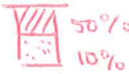
Bonus Questions

- Ivy is working on a chemistry experiment that involves acids. The first chemical solution has 10% of an acid. The second chemical solution has 50% of the same acid.
  - In what proportions should we mix the two solutions in order to get a 30% solution? (Hint: Draw a number line!)

we use the number line to find a ratio:

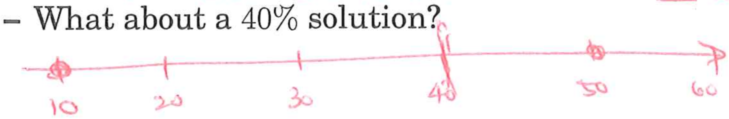


Since a, b are equal, then we want the ratio of our solution to be 1 part 10% acid and 1 part 20 % acid.

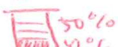
To check: our acid looks like: 

so $\frac{1}{2}\times 50\% + \frac{1}{2}\times 10\% = 25\% + 5\% = 30\%$

  - What about a 40% solution?



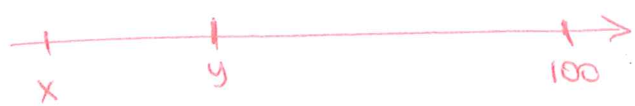
From the number line, we know we want 3 parts of 1 acid and 1 part of another acid. Since 40% is closer to 50%, we want there to be more 50% acid. So 3 parts 50% acid and 1 part 10% acid.

To check: 

$\frac{1}{4}\times 10\% + \frac{3}{4}\times 50\% = 2.5\% + 37.5\% = 40\%$.

- A gold and copper alloy has x% of gold. How much gold do you have to add to the alloy to get an alloy with y% of gold?

Pure gold is 100% so our numberline looks like.



$y-x$ is the parts pure gold we want to add and $100-y$ is the parts of the x% alloy we want to add. So gold : x% alloy :: $y-x$: $100-y$