

# Permutations: Part 1

Junior Circle

11/1/2015

## Post-Halloween Warm-Ups

1. Find any solution to the following cryptarithm.

$$\begin{array}{r} G H O S T \\ + G H O S T \\ \hline H O U S E \end{array}$$

2. Four teachers will be doing face painting. Each teacher can paint one child's face in three minutes.

(a) How many faces will all four teachers be able to paint in an hour?

(b) In three hours?

3. Two groups of trick-or-treaters go into a neighborhood with a row of 51 houses. One group starts at the end house on the left and visits every 6th house while the other group starts at the end house on the right and visits every 8th house. Will they ever meet? If so, at which house?

4. You have 3 Tootsie Rolls, 8 pieces of candy corn, and 5 apple-flavored Jolly Ranchers in your Trick-or-Treat bag.

(a) If you reach into the bag, what is the probability you will select a Tootsie Roll?

(b) How many candies must you pick to make sure that you have at least one candy corn and one Jolly Rancher?

5. Witches and wizards came to a Halloween party. Each of 5 wizards gave a spider to 8 of the witches. As a result, every witch got 4 spiders. How many witches came to the party?

6. Three wizards drink a bucket of brew in 4 hours. How many wizards will it take to drink the bucket in 12 hours?

## Permutations and Combinations

A permutation of a set of objects is an arrangement of those objects in a particular order. The order of the objects matters. In the question below, we are permuting the order the marbles sit in.

1. How many ways are there to put three marbles of 3 different colors in a row?

**Definition 1** *The product of all the natural numbers from 1 through  $n$  is called  $n$  factorial.*

$$n! = 1 \times 2 \times 3 \times \dots \times n$$

For example,  $3! = 6$ . It is a useful convention to set  $0! = 1$ .

2. Compute the following:

a)  $5! =$

b)  $6! =$

3. How many ways are there to put 5 marbles of 5 different colors in a row?

4. How many ways are there to put  $n$  marbles of  $n$  different colors in a row?

5. Compute the following:

a)  $\frac{5!}{4!} =$

b)  $\frac{100!}{98!} =$

c)  $\frac{n!}{(n-1)!}$

6. There are 10 marbles of different colors in a box. How many ways are there to put 6 of them in a row? Write down the answer using the  $n!$  notation.

7. There are  $n$  marbles of different colors in a box. How many ways are there to put  $k$  of them in a row?

**Definition 2** *A way to choose  $k$  objects out of  $n$  so that the order of the chosen objects matters is called a permutation.*

As you saw in the previous problem, the number of permutations when ordering  $k$  objects out of  $n$  total objects is given by the following.

$$P(n, k) = \frac{n!}{(n - k)!}$$

### **More Problems**

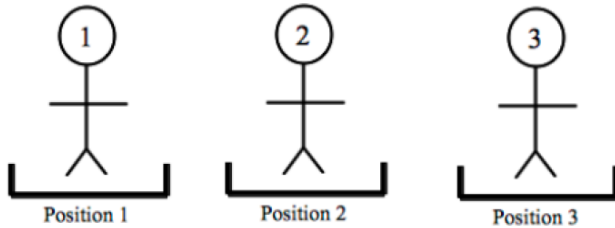
8. There are 100 marbles of different colors in a box. How many ways are there to put 2 of them in a row?

9. We want to put 4 beads on a string. How many unique ways are there to order the 5 beads on the string? (Hint: The string has no directionality. So, ABCD on the string is the same as DCBA, since you would just be looking at it from the other side.)

10. We want to put 4 beads on a bracelet. How many unique ways are there to put the 4 beads on a bracelet? (Hint: A bracelet is a continuous circle, so there is not starting bead or ending bead.)

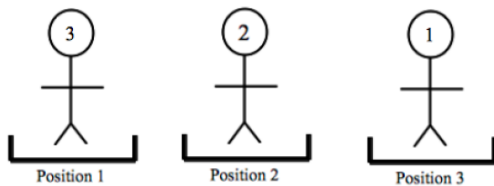
## A Deeper Look at Permutations

Several people (with labels 1, 2, 3... attached to the back of their T-shirts) are standing in positions 1, 2, 3... at the beginning:



Suppose that they switch positions. For example, let's say that

- people in positions 1 and 3 switch positions
- person in position 2 remains there



We can write this as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix}.$$

The first row represents the starting positions. The second row represents the end positions. To make the notation shorter, we will not be drawing arrows, so that the above looks like

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

(1) Suppose that we have the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

(a) What positions did not shift?

(b) Where did position 3 move to?

(c) What position moved into position 3?

(2) In a row of 5 people, the middle person stayed in his place. The two people at the ends of the row switched places. Their neighbors also switched places. Indicate where each of the positions moved to:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

(3) In a row of 5 people, all of the odd positions didn't change. The two even positions switched. Indicate where each of the positions moved to:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

(4) In a row of 6 people, the first 5 people shifted one place to the right, and the last person moved to the first place. Write this down as a permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

(5) Compare the following three permutations. Which of them would you call more “mixed-up”? Why? (Hint: Circle the positions that changed)

- The first permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix};$$

- The second permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix};$$

- The third permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix};$$

(6) The simplest possible permutation just switches two positions (and keeps the rest the same). Such permutations are called *transpositions*.

Give several examples of transpositions below:

(a)

$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Give an example of a permutation that is NOT a transposition:

(a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$



It turns out that transpositions are the building blocks of permutations. By performing several transpositions in a row, you can get various permutations. The amazing fact is that you can get any permutation this way.

(7) In a row of three people,

(a) Write down the permutation that switches positions 1 and 2;

(b) Then, write down the permutation that switches positions 2 and 3;

(c) Perform these two permutations (first a, then b) and record the end result:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}.$$

(8) In a row of three people,

(a) Write down the permutation that switches positions 2 and 3;

(b) Then, write down the permutation that switches positions 4 and 2;

(c) Finally, write down the permutation that switches positions 1 and 3:

(d) Perform these three permutations (first a, then b and finally c) and record the end result:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

- (9) How can we get the following permutation by performing several switches?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 1 & 3 \end{pmatrix};$$

- Model this with the following:
  - First, switch positions  and  ;
  - Second, switch positions  and  .
- Can you find another way?
  - First, switch positions  and  ;
  - Second, switch positions  and  .

- (10) In the previous problem, we performed one permutation followed by another permutation. Let's do something like this again.

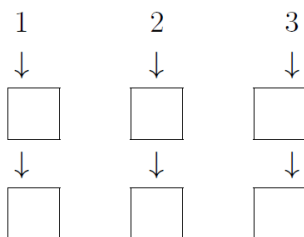
- Suppose you first perform the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix}.$$

- Then, you perform the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix}.$$

What is the final result when you perform the above two transpositions?  
Remember that the notation above refers to switching two positions, regardless of the number occupying the position.



- If we forget about the intermediate step, this can be written as:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}.$$

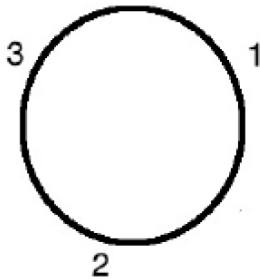
- To show the result of these two permutations performed one after the other we write:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

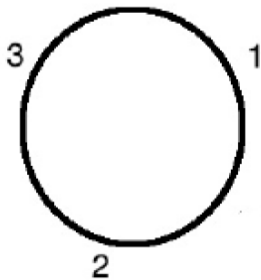
- (11) Find the result of performing the following two permutations one after the other:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow \end{pmatrix}.$$

- If 3 people are sitting around the tables, what does the first permutation represent? (Use arrows to show where the person moves.)



- What does the second permutation represent? (Use arrows to show where the person moves.)



- What is the result of doing these 2 operations one after the other?
- Does this agree with your answer above?

HOMEWORK: Combine 3 transpositions to get a permutation, as shown below. Be prepared to share with a partner next week.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

## Combinations (For those who have finished.)

**Definition 3** *A way to choose  $k$  objects out of  $n$  so that the order of the chosen objects does not matter is called a combination.*

1. There are 3 different marbles. How many ways to choose 2 out of the bag of 3?

2. There are 100 different marbles. How many ways to choose 100 out of the bag of 100?

3. There are 100 different marbles. How many ways to choose 1 out of the bag of 100?

4. There are 100 different marbles. How many ways to choose 0 out of the bag of 100?

5. There are 100 different marbles. How many ways to choose 98 out of the bag of 100?