Ordinals and Cardinals

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1   Introduction

Definition 1   A function f from a set S to a set T is a rule that to any element $s \in S$ associates an element $t = f(s) \in T$.

Exercise 1.1   .   Let $S = \{$Andersons, Browns, Cortez, Denners, Franks, Greens$\}$ be the set of all families living on Grand avenue. Let T be the same set. Suppose that f is a rule that describes which families are neighbors: Andersons live between Browns and Cortez; Cortez live next to Denners and Andersons; etc. Is f a function? Why?

Definition 2 A function $f : S \to T$ is called a bijection if

1. For each element $t \in T$ there is $s \in S$ such that $f(s) = t$.

2. If $f(s_1) = f(s_2) \in T$, then $s_1 = s_2$.

The first condition means that the range of the function is the entire set T . If a function has this property it is called onto.

The second condition means that there is a unique element in the set S that is mapped to a given element T by function f. If a function has this property it is called one-to-one.

Exercise 1.2 Give an example of a function between two finite sets which is

- not onto; Is your function one-to-one?

- not one-to- one; Is your function onto?

Exercise 1.3 Show that if a function $f : S \to T$ is a bijection, then the inverse function $f^{?1} : T \to S$ is well defined. To compare the magnitude (or size) of two sets the basic notion is that of equivalence.

Definition 3 Two sets S and T are called equivalent if there is a bijection $f : S \to T$.

Exercise 1.4 Let S and T be finite sets. Show that S and T are equivalent if and only if they have the same number of elements.

The last exercise implies that no finite set can be equivalent to its proper-subset. (A subset S of a set T is called proper if $S \subseteq T$ and $S \neq T$). In simple terms, for finite sets the whole is "greater" than any of its "parts." The situation changes radically when we start dealing with infinite sets.

2   Countable Sets

The simplest infinite set is that of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$. A set that is equivalent to $\mathbb{N}$ is called countable because its elements can be counted using $\mathbb{N}$. Modern set theory, originated by German mathematician Georg Cantor and his school, started the current mathematical study of infinity.

Exercise 2.1   Find a bijection between the set of positive even numbers $2\mathbb{N} = \{2, 4, 6, 8, \ldots\}$ and the set of all natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. This bijection establishes the equivalence of these two sets. On the other hand, the first subset is clearly a proper subset of the second one.

Exercise 2.2   Show that the set of all positive odd numbers $\{1, 3, 5, 7, 9, \ldots\}$ is countable.

Exercise 2.3   Show that the set of all integers (positive and negative) $\mathbb{Z} = \{\ldots, ?3, ?2, ?1, 0, 1, 2, 3, \ldots\}$ is countable.

Exercise 2.4   Show that the set of all ordered pairs of natural numbers, $\mathbb{N} \times \mathbb{N}$, is countable. (Hint: depict this set as a lattice of points, and try to find a path that covers the entire lattice).

Exercise 2.5   Show that $\mathbb{Q}$, the set of all rational numbers, is countable.

Theorem 1   Cantor-Schröder-Bernstein If sets S, T have functions $f : S \to T$ and $g : T \to S$ that are both one-to-one, then there is a bijection $h : S \to T$.

The proof of this theorem is a bit techincal, but can be outlined as follows: create chains of elements $s, f(s), g(f(s)), \ldots$. Every element in each of S and T is in such a chain; so map each element to the next one in the chain. A few details need to be ironed out with chains that have an endpoint in T, but that is the gist of the proof.

Exercise 2.6   Show that each of the following sets is countable. (Be sure to note when you use the Cantor-Schröder-Bernstein Theorem, and in the cases you've already proven, re-prove using CSB)

1. Ordered pairs of natural numbers

2. Ordered triples (and quadruples, etc.) of natural numbers

3. Finite (ordered) sequences of natural numbers

4. Finite (unordered) sets of natural numbers

5. Rational numbers

6. Polynomials with rational coefficients

7. Algebraic numbers (that is, complex numbers that are solutions to polynomials with real integer coefficients)

3 Cardinal Arithmetic

Definition 4   A *cardinal number* is the size of a set; two sets have the same cardinality, $|X| = |Y|$ if there is a one-to-one and onto function $f : X \to Y$.

Definition 5   The ordering of cardinal number is given by saying $|X| \leq |Y|$ if there is a one-to-one function $f : X \to Y$.

Exercise 3.1   Show that the ordering of cardinal numbers is:

1. Reflexive ($|X| \leq |X|$)

2. Transitive (if $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$)

3. Antisymmetric (if $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$ - hint, you can use a theorem from last week for this part)

Definition 6   If $|X| \leq |Y|$ and $|X| \neq |Y|$, we say $|X| < |Y|$

Definition 7   For any set X, the *powerset* of X, denoted $\mathcal{P}(X)$, is the set of all subsets of X.

Exercise 3.2   Write down the powersets of the following:

- $\{a, b\}$

- $\{\}$

- $\{1, 2, 3\}$

Theorem 2   Cantor's Theorem
   For every set X, $|X| < |\mathcal{P}(X)|$.

Exercise 3.3   Prove Cantor's Theorem:

1. First, show that $|X| \leq |\mathcal{P}(X)|$ by finding a one-to-one function from X to $\mathcal{P}(X)$.

2. For any function $f : X \to \mathcal{P}(X)$, consider the set $Y = \{x \in X : x \notin f(x)\}$. Could any $a \in X$ have $f(a) = Y$?

3. Explain why this means that $|X| \neq |\mathcal{P}(X)|$, and thus $|X| < |\mathcal{P}(X)|$.

   Now that we have cardinal numbers, we can define arithmetic operations between them. We'll use greek letters to indicate cardinal numbers.

Definition 8   1. $\kappa + \lambda = |A \cup B|$, where $\kappa = |A|, \lambda = |B|$, and $A \cup B$ is the set containing all the elements of both A and B.

2. $\kappa \cdot \lambda = |A \times B|$, where $\kappa = |A|, \lambda = |B|$, and $A \times B$ is the set of ordered pairs of elements, the first from A, the second from B.

3. $\kappa^\lambda = |A^B|$, where $\kappa = |A|, \lambda = |B|$, and A^B is the set of all functions from B to A.

Exercise 3.4   Explain why the arithmetic operations on cardinals, when applied to finite numbers, are the same as normal addition, multiplication, and exponents.

Exercise 3.5   Show that if $|A| = \kappa$, then $|\mathcal{P}(A)| = 2^\kappa$ (Hint: for any subset X of A, consider the function $\chi_X : A \to \{0, 1\}$ such that $\chi_X(x) = 1$ if $x \in X$, and $\chi_X(x) = 0$ if $x \in A - X$).

4 Uncountable Sets

Theorem 3   Cantor's Diagonalization Argument The set $\mathbb{R}$ of real numbers is not countable.

To see why this is true, suppose we could write down a countable list of all the real numbers in order:

$a_1.b_1c_1d_1\ldots$
$a_2.b_2c_2d_2\ldots$
$a_3.b_3c_3d_3\ldots$

Then, you can construct a real number $a_\infty.b_\infty c_\infty d_\infty$ by choosing $a_\infty \neq a_1$, $b_\infty \neq b_2$, $c_\infty \neq c_3$, and so on. This number isn't on the list, contradicting the original assumption of a bijection $\mathbb{N} \to \mathbb{R}$.

Exercise 4.1   Show that each of the following sets has the same cardinality as the real numbers. (Again, note where you use Schröder-Bernstein.)

1. Countable sequences of 0's and 1's

2. Sets of natural numbers

3. Ordered pairs of real numbers

4. Ordered triples (and quadruples, etc.) of real numbers

5. Finite sequences of reals

6. Complex numbers

7. Polynomials with complex coefficients

8. Countable sequences of reals

9. Countable sequences of natural numbers

10. Irrational numbers (Hint: consider continued fraction expansions)

11. Functions from the rationals to the reals

12. Continuous functions from the reals to the reals

Exercise 4.2   Show that each of the following sets has larger cardinality than the reals.

1. Sets of real numbers

2. Real-valued (not necessarily continuous) functions of real numbers

3. Uncountable sets of reals

5   More Arithmetic

Exercise 5.1   Show that if X and Y have the same cardinality, then $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ have the same cardinality.

Exercise 5.2   By describing appropriate one-to-one functions, show that the following hold for any cardinal numbers κ, λ, μ:

1. $(\kappa \cdot \lambda)^{\mu} = \kappa^{\mu} \cdot \lambda^{\mu}$

2. $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$

3. $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}$

4. If $\kappa \leq \lambda$, then $\kappa^{\mu} \leq \lambda^{\mu}$

5. If $0 < \lambda \leq \mu$, then $\kappa^{\lambda} \leq \kappa^{\mu}$.

6. $1^{\kappa} = 1$