Induction II

Math Circle

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Last time we learned what induction is, but just in case you need a refresher, every inductive proof has 2 steps. Suppose that you have a statement $P(n)$, which might, or might not, be true and depends on a a number n. An example is the statement that:

$$1 + 2 + 3 \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

If we wanted to prove that the statement $P(n)$ was true for all n, then we can use induction. Every inductive proof has 2 steps.

Step 1 This is called the base case. Show that the statement $P(n)$ is true, for some small, starting value of n, for example show that $P(1)$ is true.

Step 2 This is the inductive step. In this one, you assume that you knew for sure that $P(n)$ was true. Can you use the fact that $P(n)$ is true to show that $P(n+1)$ is also true?

As an example, lets show that the statement

$$1 + 2 + 3 \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

is true for every n.

Step 1 Is $P(1)$ true? Well, $P(1)$ is that $1 = \frac{1(1+1)}{2}$, which is definitely true because $1 = \frac{2}{2}$. On to step 2!

Step 2 If $P(n)$ is true, does that automatically mean that $P(n+1)$ is true? Well, if we start with $P(n)$ we have that $1 + 2 + 3 \cdots + (n-1) + n = \frac{n(n+1)}{2}$ is true. But we can add $n+1$ to both sides and get

$$1 + 2 + 3 \cdots + (n-1) + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

If we get a common denominator on the right hand side, and multiply out $n(n+1)$ we get:

$$1 + 2 + 3 \cdots + (n-1) + n + (n+1) = \frac{n^2 + 3n + 2}{2}$$

But we can factor $n^2 + 3n + 2$ to get:

$$1 + 2 + 3 \cdots + (n-1) + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

Which is exactly $P(n+1)$! So we showed that if you know that $P(n)$ is true, then $P(n+1)$ must also be true. This completes the inductive step, which proves that the statement $P(n)$ is true whenever $n = 1, 2, 3, \ldots$

Now it's your turn. Try and use induction to show that the following formulae are valid:

1. Use the hints $(n+1)^2 = n^2 + 2n + 1$ and $n(n+1)(2n+1) = n^3 + 3n^2 + n$ and $n^3 + 9n^2 + 13n + 6 = (n+1)(n+2)(n+3)$ to show that

   $$1^2 + 2^2 + 3^2 \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

   is true for all $n = 1, 2, 3, \ldots$

2. Use the hints that $n(n+1)(n+2) = n^3 + 3n^2 + 2n$, and $(n+1)(n+2) = n^2 + 3n + 2$ and $n^3 + 6n^2 + 11n + 6 = (n+1)(n+2)(n+3)$ to show that

   $$1 \cdot 2 + 2 \cdot 3 + \ldots (n-1) \cdot n + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

   is true for all $n = 1, 2, 3, \ldots$

3. Prove or disprove: For any x which is a natural number, $x^2 + x + 41$ is always a prime number.

4. Prove that:

   $$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \ldots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

5. **Challenge 1** Here are a few more if you are looking for a challenge! Hint, remember that $(a+b)^2 = a^2 + 2ab + b^2$ and try using one of the previous formulae!

   $$(1 + 2 + \cdots + (n-1) + n)^2 = 1^3 + 2^3 + \cdots + (n-1)^3 + n^3$$

6. **Challenge 2** And one more, if you are feeling saucy. Let F_n be the n'th Fibonacci number, so $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Prove that

   $$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$$

   for $n > 2$.

   Ok, ok, enough of working with stale formulas. Let's draw some pictures!

7. Suppose that you work at Clear Skies Ceramics, a factory that manufactures beautiful azure blue ceramic tiles. Your factory can make these tiles in two sizes, either in a square 1 inch by 1 inch, or in a rectangle that is 1 inch by 2 inch.

   People love your tiles, because then add a decorative flair to their bathroom trim! Suppose that one section of trim is 1 inch by 2 feet long. How many different ways can you tile the trim with your with identical 1 by 1 and 1 by 2 tiles? (Hint: Start with a smaller trim that's 1 inch by n inches for $n = 1, 2, 3, \ldots$ and see if you notice a pattern.)

8. Prove that if release a new line of tiles that are formed by gluing together one of your 1 by 1 and one of your 1 by 2 tiles into an L shape, then you can tile any wall that is 2^n by 2^n if you don't have to tile the top right 1 inch by 1 inch spot.

9. You are playing a game with your friend. Your friend puts n blue and n red chips in a circle, and then you pick a starting chip, and remove each chip in the circle going counterclockwise. You win if you can remove every chip and there are never, ever more blue chips than red chips that haven't been removed. Can you always win? If so, prove it. If not, give a counterexample.

10. Prove that if you cut the plane into pieces using a bunch of nonparallel lines, and you never have 3 lines intersect at the same point, than you can always color each piece either red or blue, such that a blue (or red) piece is never touching another blue (or red) piece on it's side.

11. If $8^n - 1$ always divisible by 7? If so prove it, if not find a counterexample.

12. Show that 9 always divides $4^n + 15n - 1$ for any natural number n.

13. What's wrong with the following proof that all horses are the same color?

    If there was just one horse, than certainly every horse is the same color.

    If you have more than one horse, say n of them, line them up end to end. By the inductive step, horses $1, 2, \ldots, n-1$ are the same color, and so are horses $2, 3, \ldots, n$, so all horses must be the same color.

14. Suppose that you have a grid that is 5 by 7 tiles. Your goal is to figure out how many different paths there are that connect the top left, and the bottom right of the grid if you are only allowed to move one tile to the right, or one tile down. Can you find a formula that works if you grid is n by m tiles?