Math Circle
Beginners Group
October 25, 2015
Solutions

Warm-up problem

1. One hundred persons are lined up in a single file, facing north. Each person has been assigned either a yellow hat or a green hat. No one can see the color of his or her own hat. However, each person is able to see the color of the hat worn by every person in front of him or her. For example, the last person in line can see the colors of the hats on 99 persons in front of him or her; the first person, who is at the front of the line, cannot see the color of any hat.

   Beginning with the last person in line, and then moving to the 99th person, the 98th, etc., each person is asked to name the color of his or her own hat. If the person correctly names the color of the hat, he or she lives. If the color is incorrectly named, the person is shot dead on the spot.

   Everyone in the line is able to hear every response as well as the gunshot. Also, everyone in the line is able to remember all that needs to be remembered and is able to compute all that needs to be computed.

   Before being lined up, the 100 persons are allowed to discuss strategy, with an eye toward developing a plan that will allow as many of them as possible to correctly name the color of their hat and thus, survive. Once lined up, each person is allowed to say only "yellow" or "green" when his or her turn arrives, beginning with the last person in line.

   Your assignment: Develop a plan that allows as many people as possible to live. Do not waste time attempting to evade the stated bounds of the problem; there is no trick to the answer.[1]

   > You have a 100% chance of saving all but the last prisoner, and a 50% chance of saving the last one. Here is the strategy the prisoners should agree on.
   >
   > The last prisoner counts the number of green hats in front of him. If the number is even, he yells "green." If the number is odd, he yells "yellow."
   >
   > If the 99th prisoner hears "green," but counts an odd number of green hats in front of him, then he knows that his hat must be green so that the total number of green hats can be even. If he counts an even number of green hats in front of him, then his hat must be yellow.

[1] The Puzzle of 100 Hats and the solution have been sourced from The New York Times.

If the last person yells "yellow," then the 99th prisoner knows that there are an odd number of green hats. So, he counts the number of green hats that he sees in front of him. Again, if they are odd, he knows that his hat is yellow. If they are even, his hat must be green. The 99th prisoner then yells out the color of his hat and is spared.

The next prisoner takes into account whether the 99th prisoner had a green hat or not. He now knows whether the remaning number of green hats, including his own, is odd or even. Then by counting the number of green hats that he sees, he knows the color of his own hat. So he yells out the color of his hat and is spared.

This saves all but the last prisoner, and there is a 50% chance that his hat is the color he shouted out.

(Students should start thinking about the problem with a fewer number of prisoners.)

What happens if there are n prisoners?

The same logic applies with n prisoners. You have a 100% chance of saving $n-1$ prisoners, and a 50% chance of saving the nth prisoner.

2. What do you think $\max(n, m)$ means? What is $\max(2, 3)$?

This is the maximum function. It returns the largest value from the given set of values.

$\text{Max}(n, m) = n$, if $n > m$.

$\text{Max}(n, m) = m$, if $n < m$.

$\text{Max }(2, 3) = 3$

Lengths of Terminating Decimals and Lengths of Periods

1. How can one find the length of a terminating decimal?

   (a) Let $\frac{a}{b}$ be a fraction in its simplest form. If $\frac{a}{b}$ represents a terminating decimal, we know that

   $$b = 2^n \cdot 5^m$$

   (b) Apart from using long division, how can you convert $\frac{a}{b} = \frac{a}{2^n \cdot 5^m}$ into a decimal? (Hint: Multiply the numerator and denominator by an appropriate number.)

   i. Suppose that $n = m$.

   $$\frac{a}{b} = \frac{a}{2^n \cdot 5^n} = \frac{a}{10^n} = 0.\underline{a_1}\,\underline{a_2}\ldots\underline{a_n}^2$$

   ii. Suppose that $n > m$.

   $$\frac{a}{b} = \frac{a}{2^n \cdot 5^m} \times \frac{5^{n-m}}{5^{n-m}} = \frac{5^{n-m} \cdot a}{2^n \cdot 5^n} = \frac{5^{n-m} \cdot a}{10^n} = 5^{n-m}(0.\underline{a_1}\,\underline{a_2}\ldots\underline{a_n})$$

   iii. Suppose that $n < m$.

   $$\frac{a}{b} = \frac{a}{2^n \cdot 5^m} \times \frac{2^{m-n}}{2^{m-n}} = \frac{2^{m-n} \cdot a}{2^m \cdot 5^m} = \frac{2^{m-n} \cdot a}{10^m} = 2^{m-n}(0.\underline{a_1}\,\underline{a_2}\ldots\underline{a_m})$$

   (c) How many digits are in the decimal expansion of $\frac{a}{b}$ if

   i. $n = m$?

   From part (b) (i), there are n digits in the decimal expansion of $\frac{a}{b}$.

   ii. $n > m$?

   From part (b) (ii), there are n digits in the decimal expansion of $\frac{a}{b}$.

   iii. $n < m$?

   From part (b) (iii), there are m digits in the decimal expansion of $\frac{a}{b}$.

   (d) Finally, what is the length of the terminating decimal corresponding to the fraction $\frac{a}{2^n \cdot 5^m}$?

   The length of the terminating decimal corresponding to the fraction $\frac{a}{2^n \cdot 5^m}$ is $\max(n, m)$.

[2] The digits are underlined to distinguish the number $0.\underline{a_1}\,\underline{a_2}\ldots\underline{a_n}$ written with digits $a_1, a_2, \ldots, a_n$ from the product of the numbers $a_1 a_2 \ldots a_n = a_1 \cdot a_2 \cdot \ldots \cdot a_n$.

2. Find the lengths of the decimal expansions of the following fractions.

   (a) $\frac{3}{2^5 \cdot 5^3}$

   Max$(5,3) = 5$

   So, the length of the decimal expansion is 5.

   (b) $\frac{17}{1250}$

   $\frac{17}{1250} = \frac{17}{625 \times 2} = \frac{17}{5^4 \times 2}$
   Max$(4,1) = 4$

   So, the length of the decimal expansion is 4.

3. Sam says, "When you convert a fraction into a periodic decimal, there are only 10 possible digits in the period. Once you have used them all up in any order, you have to repeat them. So, the repeating section starts over. This means the length of the period of any repeating decimal you get from a fraction is at most 10." Is he right? Why or why not?

   (a) When you convert a fraction into a decimal using long divison or the quotients method we used last week, what (if anything) forces the decimal expansion to eventually repeat itself? (Hint: The next step in the process of conversion is determined by the *remainder*.)

   There are only a limited number of possibilities for the remainder after you divide by the denominator. Since the remainder completely determines the next line, once a remainder repeats, the whole process repeats.

   (b) So, can there be a decimal expansion in which the length of the period is more than 10? Why or why not?

   We get each new decimal digit as the quotient when we divide a multiple of 10 by the denominator. But it is possible for two different multiples of 10 to have the same quotient when divided by the denominator (if the denominator is greater than 10), but different remainders. If this happens, we'll get the same new digit in both cases, even though the remainders are different and the process of repetition has not started yet. (For example, if we compute $\frac{5}{22}$, the first two digits are 2, but $\frac{5}{22} \neq 0.\overline{2}$.)

(c) Without calculating directly, how long could the period of the decimal expansion of the fraction $\frac{1}{17}$ be?

We cannot have a period longer than 17 itself, since there are only 17 different possible remainders when you divide by 17. Moreover, we can say that the period cannot be longer than 16. This is because if the remainder 0 ever occurs, then each line from then on will be $0 = 0 \times 17 + 0$. This corresponds to a terminating decimal (which we can think of as having period 1 because of the repetition of 0). And if 0 never occurs, then there are only 16 possible remainders which may occur, so the maximum length of the period is 16.

(d) What is the maximum possible length of period of a number of the form $\frac{1}{n}$?

$n - 1$

Pure and Mixed Periodic Decimals

Periods in decimals can start either immediately after the decimal point or not.

In $0.212121... = 0.\overline{21}$, the period 21 starts immediately after the decimal point. These decimals are called **pure periodic decimals**.

However, in $0.124444... = 0.12\overline{4}$, the period 4 does not immediately start after the decimal point. Such fractions are called **mixed periodic decimals**.

1. Convert the decimal $0.\overline{124}$ into a fraction and find the prime factorization of its denominator.

   Let $w = 0.\overline{124}$
   $1000w = 124.\overline{124}$
   $w = 0.\overline{124}$
   $1000w - w = 124.\overline{124} - 0.\overline{124} = 124$
   $999w = 124$
   $w = \frac{124}{999}$

2. To convert $0.12\overline{4}$ into a fraction, write it as

$$0.12\overline{4} = 0.12 + 0.00\overline{4}.$$

   (a) Convert 0.12 (the part of the decimal before the period) into a fraction.

   $0.12 = \frac{12}{100} = \frac{3}{25}$

   (b) Convert $0.00\overline{4}$ into a fraction as follows. Notice that $0.00\overline{4} = \frac{1}{100} \cdot 0.\overline{4}$. Convert $0.\overline{4}$ into a fraction and then use this result to convert $0.00\overline{4}$.

   Let $x = 0.\overline{4}$
   $10x = 4.\overline{4}$
   $10x - x = 4.\overline{4} - 0.\overline{4} = 4$
   $9x = 4$
   $x = \frac{4}{9}$
   Therefore, $0.00\overline{4} = \frac{1}{100} \cdot x = \frac{1}{100} \cdot \frac{4}{9} = \frac{4}{900} = \frac{1}{225}$.

   (c) Combine results in (a) and (b) to find the fraction representation of $0.12\overline{4}$.

   $0.12\overline{4} = 0.12 + 0.00\overline{4}$
   Therefore, $0.12\overline{4} = \frac{3}{25} + \frac{1}{225} = \frac{27+1}{225} = \frac{28}{225}$.

(d) Explain why the period in the decimal representation of this number does not start right away. (Hint: Think about the prime factorization of the denominators of the fractions you obtained in (a) and (b).)

The decimal is composed of a terminating part and a periodic part. This is why the period does not start right away. We can observe that the prime factorization of 225 is $5^2 \times 3^2$. If there is a factor of 5 or 2 in the denominator coming from the terminating part of the decimal, the period does not start right away in the decimal representation.

3. Convert the following periodic decimals into fractions.

(a) $0.633... = 0.6\overline{3}$

$0.6\overline{3} = 0.6 + 0.0\overline{3}$
We will convert 0.6 and $0.0\overline{3}$ into decimals separately.
$0.6 = \frac{6}{10} = \frac{3}{5}$
Let $y = 0.0\overline{3}$
$10y = 0.\overline{3}$
$100y = 3.\overline{3}$
$100y - 10y = 3.\overline{3} - 0.\overline{3} = 3$
$90y = 3$
$y = \frac{3}{90} = \frac{1}{30}$
Therefore, $0.6\overline{3} = \frac{3}{5} + \frac{1}{30} = \frac{18+1}{30} = \frac{19}{30}$.

Here is an easier way to solve this problem without splitting the two parts of the decimal.
Let $z = 0.6\overline{3}$
$10z = 6.\overline{3}$
$100z = 63.\overline{3}$
$100z - 10z = 63.\overline{3} - 6.\overline{3} = 57$
$90z = 57$
$z = \frac{57}{90} = \frac{19}{30}$
Therefore, $0.6\overline{3} = z = \frac{19}{30}$.

(b) $0.173434... = 0.17\overline{34}$

$0.17\overline{34} = 0.17 + 0.00\overline{34}$
We will convert 0.17 and $0.00\overline{34}$ into decimals separately.
$0.17 = \frac{17}{100}$
Let $y = 0.00\overline{34}$
$100y = 0.\overline{34}$
$10000y = 34.\overline{34}$
$10000y - 100y = 34.\overline{34} - 0.\overline{34} = 34$
$9900y = 34$
$y = \frac{34}{9900}$
Therefore, $0.17\overline{34} = \frac{17}{100} + \frac{34}{9900} = \frac{1683+34}{9900} = \frac{1717}{9900}$.

Here is an easier way to solve this problem without splitting the two parts of the decimal.
Let $z = 0.17\overline{34}$
$100z = 17.\overline{34}$
$10000z = 1734.\overline{34}$
$10000z - 100z = 1734.\overline{34} - 17.\overline{34} = 1717$
$9900z = 1717$
$z = \frac{1717}{9900}$
Therefore, $0.17\overline{34} = z = \frac{1717}{9900}$.

(c) 0.12312341231234...

Let $x = 0.\overline{1231234}$
$10000000x = 1231234.\overline{1231234} = 10^7x$
$10^7x - x = 1231234.\overline{1231234} - 0.\overline{1231234} = 1231234$
$(10^7 - 1) \cdot x = 1231234$
$x = \frac{1231234}{10^7-1}$

Rational and Irrational Numbers

1. The decimal number 0.1234567891011121314151617181920 21... is formed by writing down the digits of subsequent whole numbers. Can you convert it into a fraction? Why or why not?

   All fractions represent periodic decimals. This decimal number is not periodic, so we cannot convert it into a fraction.

   A **rational number** is a number which can be written in the form $\frac{a}{b}$, where a and b are both integers and $b \neq 0$.

   An **irrational number** is one which cannot be expressed as $\frac{a}{b}$, where a and b are both integers and $b \neq 0$. (It is not a number that is crazy!)

2. The decimal representation of a rational number is always <u>*periodic*</u>.

3. Is the reverse also true? If a number has a decimal representation which is <u>*periodic*</u>, does it mean that the number must be rational?

   Yes.

4. Are the following numbers rational? Why or why not?

   (a) 35

   Yes, it can be written as a ratio of two integers: $\frac{35}{1}$.

   (b) -57.32

   Yes, it can be written as a ratio of two integers:$-\frac{5732}{100}$.

   (c) 0.1010010001...

   No, it cannot be written as a ratio of two integers since the decimal number is not periodic.

   (d) 0

   Yes, it can be written as a ratio of two integers: $\frac{0}{1}$.

(e) $\frac{2.735687356873568...}{3}$

Yes, it can be written as $\frac{2.\overline{73568}}{3}$, and we know that $2.\overline{73568}$ can be represented as a fraction. You do not need to find the fraction, but know that $\frac{2.735687356873568...}{3}$ can thus be written as a ratio of two integers.

(f) $\frac{\frac{13}{17^2} \cdot 2^2 \cdot \frac{14}{9}}{55^3 \cdot 19}$

You do not need to simplify the fraction, but identify that it is composed of ratios of integers. It can thus be simplified as just a ratio of two integers, and is a rational number.

5. Why does the product of a rational number and a periodic decimal result in a periodic decimal?

   A periodic decimal can be represented as a fraction. The product of a fraction and a rational number will result in a rational number. The decimal representation of this rational number would be periodic.

6. Can you give another example of irrational numbers?

   Any decimal number that is not periodic is an irrational number. For example, 0.0204081632641282565 12...

Pi (π) is a popular example of irrational numbers.

$$\pi = 3.14159265358979323846264338327 95...$$

There is a (very hard) proof that demonstrates that π is irrational.

7. Is there a fraction that equals to π?

   Since π is an irrational number, there is no fraction or ratio of integers that equals π. One might think of $\frac{22}{7}$, but $\frac{22}{7} = 3.1428$ is only an approximation of π.

8. Another popular example of an irrational number is $\sqrt{2}$. Let us prove that $\sqrt{2}$ is an irrational number.

   (a) We will argue by contradiction to show that $\sqrt{2}$ cannot be expressed as a ratio of two integers. So, we assume that $\sqrt{2}$ is a rational number. Then in its simplest form, it can be written as

   $\sqrt{2} = \frac{a}{b}$, where a and b do not have any common prime factors.

   (b) Rewrite this equality using cross-multiplication.

   $\sqrt{2}b = a$

   (c) Square both sides of the equality.

   $2b^2 = a^2$

   (d) Is a^2 odd or even? Is a odd or even?

   We can see from the equality above that 2 is a multiple of a^2. Therefore, a^2 is even. Since a^2 is even, a must be even.

   (e) Why is a^2 divisible by 4?

   Since a is even, it must be divisible by 2. Therefore, a^2 must be divisible by 4. (If $a = 2k$, $a^2 = 4k^2$.)

   (f) If a^2 is divisible by 4, how do we know that b^2 is divisible by 2?

   For the equality $2b^2 = a^2$ to be true, we know that 2 must be a multiple of b^2. Only then can a^2 be divisible by 4.

   (g) Is b^2 odd or even? Is b odd or even?

   Since b^2 is divisible by 2, b^2 must be even. Since b^2 is even, b must be even.

   (h) How does this contradict our original assumption?

   According to our original assumption, a and b did not have any common prime factors. However, as proven above, they are both even and have a common prime factor of 2. This contradicts our assumption.

   (i) What is your conclusion?

   We now know that $\sqrt{2}$ must be irrational, since it cannot be written in the smallest form $\frac{a}{b}$, where a and b do not have any common prime factors.

9. Here is a visual method to prove that $\sqrt{2}$ is irrational.

   (a) Once again, we will argue by contradiction. Let us assume that $\sqrt{2}$ is a rational number. So, in its simplest form, it can be written as

   $$\sqrt{2} = \frac{a}{b}$$

   Remember that a and b are the smallest possible numbers for which this equality is true.

   (b) Rewrite this equality using cross-multiplication.

   $\sqrt{2}b = a$

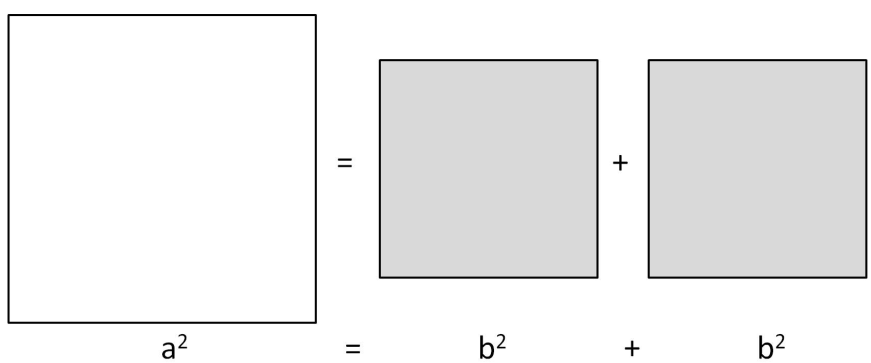
   (c) Square both sides of the equality.

   $2b^2 = a^2$

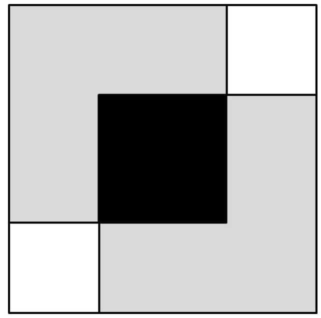
   (d) Using the equality above, express a^2 as a sum of two terms.

   $a^2 = b^2 + b^2$

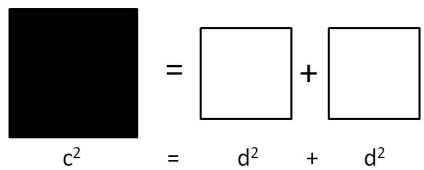
   (e) Geometrically, this means that there is a square of side a (the white square below) whose area is equal to the sum of the areas of two squares of side b (the grey squares below). Remember that we have assumed that $a \times a$ and $b \times b$ are the smallest possible squares.



(f) Since $a^2 = b^2 + b^2$, the grey squares must fit completely inside the white square. When we try to do that, the grey squares overlap in the black square, as shown below.



(g) Therefore, the area of the black square must be equal to the sum of the areas of the uncovered white corner squares.



The black square and the small white squares have integer sides, say c and d. Once again, we have found two numbers c and d such that $c^2 = d^2 + d^2$.

(h) How does this contradict our assumption?

According to our original assumption, $a \times a$ and $b \times b$ were the smallest possible squares to satisfy the equality $a^2 = b^2 + b^2$. However, as proven above, smaller squares of the dimensions $c \times c$ and $d \times d$ exist. This contradicts our assumption.

(i) What is your conclusion?

We now know that $\sqrt{2}$ must be irrational, since it cannot be written in the smallest form $\frac{a}{b}$.

10. Can you write down two numbers a and b, such that neither a nor b is rational, but $(a + b)$ is?

Yes, we can.
Let $a = \sqrt{2}$ and $b = -\sqrt{2}$. Since $\sqrt{2}$ is irrational, $-\sqrt{2}$ must be irrational.
Therefore, neither a nor b is rational.
However, $(a + b) = \sqrt{2} - \sqrt{2} = 0$, which is rational.