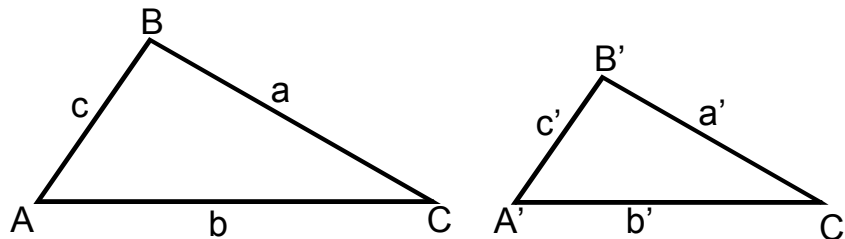# Similar Triangles

The concept of similar triangles forms an important foundation for trigonometry, but it also can be used to solve geometric problems when a trigonometric method may be very difficult or impossible to use.

The vertices or angles and the sides of one triangle can always be paired up with the vertices and sides of a second triangle (since all triangles, by definition, have exactly three vertices and exactly three sides).  To show how these are intended to be matched, it is customary to use the same letters, but with primes (a lot like apostrophes) on the symbols for one of the triangles.  Thus,



For the two triangles sketched above, the labelling of vertices and sides indicates that

angles A and A' are **corresponding angles**
angles B and B' are corresponding angles
angles C and C' are corresponding angles

and

sides a and a' are **corresponding sides**
sides b and b' are corresponding sides
sides c and c' are corresponding sides

Then, two actual triangles are said to be **similar triangles** if their sides and angles can be labelled as shown above, and then the following two properties are true:

**property (i)**:  corresponding angles are equal
(A = A' <u>and</u> B = B' <u>and</u> C = C')

**property (ii)**:  corresponding sides have proportional lengths
$$\left( \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} \right)$$

If property (i) is true, then you are guaranteed that property (ii) will be true.  If property (ii) is true, then you are guaranteed that property (i) will be true.

Notationally, if   ABC is similar to   A'B'C', then we write

ABC      A'B'C'

(Recall that we use the symbol ' ' to mean 'triangle', and that triangles are often identified by list the symbols for their three vertices.  Thus   ABC denotes the triangle with vertices A, B, and C.)
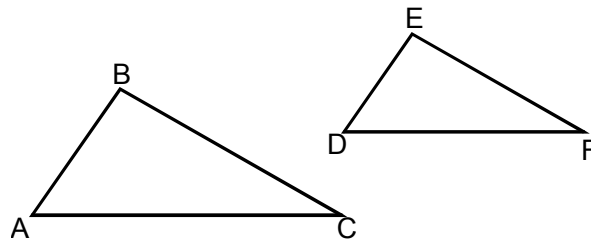
It is not necessary to use this "primed" notation, but your work in solving problems involving similar triangles must indicate explicitly how you are matching up the vertices in the two triangles.

By property (i), similar triangles have the same shape – they just differ in overall size.  For instance, if you were to draw a triangle on a transparency sheet for an overhead projector, then moving the projector towards the screen or away from the screen would produce a sequence of triangular shapes on the screen.  When the projector is close to the screen, the image would be quite small.  When the projector is further away from the screen, the image of the triangle would be larger.  However, the size of the angles forming the three vertices of the triangles would always be the same – even though their sizes were different, all of the triangle images resulting from that the triangle drawn on the transparency would have the same shape.  Hence they would all be similar to each other.  Thus, similar triangles are triangles that have the same shape, but not necessarily the same size.

The most useful application of similar triangles is in exploiting property (ii) to calculate unknown lengths which are sides or parts of sides of one triangle that is similar to another triangle for which we know corresponding lengths.  Before property (ii) can be used, however, we must use property (i) to verify that the two triangles really are similar.

**Methods Using Angles to Demonstrate That Two Triangles are Similar**

Suppose we have two triangles now labelled as



where the intention is to consider

       A to correspond to D
       B to correspond to E
and
       C to correspond to F.

Then, to demonstrate that   ABC      DEF, we must be able to demonstrate that

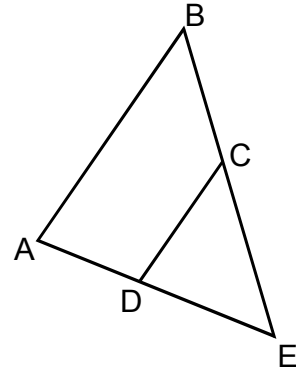       A = D,  B = E, <u>and</u>  C = F.

All three conditions must be satisfied (or true) if we wish to assert that   ABC      DEF.

We do get one small break here – if we can demonstrate that any two of these three conditions are true, then the third one is always automatically true.  Since the three angles of any triangle must add up to $180^0$, then if two triangles have two of their angles in common, the third angles must be the same in both triangles.

The two most valuable strategies for proving the equality of corresponding angles in different triangles are:

**(i) The same actual vertex belongs to both triangles.**  For example, in the sketch to the right are the triangles ABE and DCE. Obviously angle E is common to both triangles, and so we can conclude immediately that there is an angle in ABE which is equal to an angle in DCE.

**(ii) There are parallel lines in the situation whose properties can be exploited.**  For instance, in the sketch above to the right, if AB is parallel to CD, then, line BE acts as a transversal across the two parallel lines.  This means that B = C, since these two angles are corresponding angles.  Similarly, line AE also acts as a transversal across the two parallel lines, and so A = D, because these two angles are also corresponding angles.

So, for the situation shown in the sketch above, with line AB parallel to line CD, we can say the following about ABE and DCE.

> make A correspond to D
> make B correspond to C
> make E correspond to E

Now,

> E = E, because they are both the same angle (if you want some jargon, say E = E 'by
> identity')

> A = D, because A and D are corresponding angles when line AE is regarded as a
> transversal crossing the parallel lines AB and DC.

Because of these two conclusions, we are guaranteed that B = C.  Thus the three pairs of corresponding angles are each equal, and so by property (i) of similar triangles, we can conclude that

> ABE ~ DCE.

**Example:**  In the figure to the right, lines AC and DE are vertical, and line CD is horizontal.  Demonstrate that
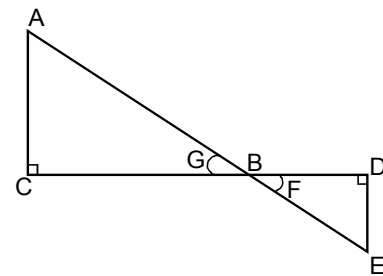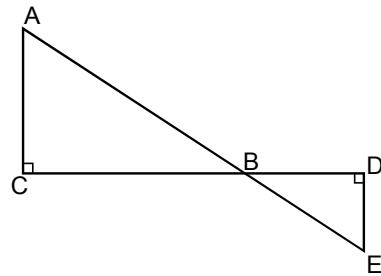
> ABC ~ EBD

**solution:**

There are two angles of interest near point B, so we need some additional labels to distinguish them, as is done in the second diagram.  When we talk about ABC, B will stand for the angle labelled G in this second diagram. Similarly, when we talk about EBD, B will stand for the angle labelled F in this second diagram.

Now, make the following correspondences:

> A corresponds to E
> C corresponds to D
> B(G) corresponds to B(F)

Clearly, C = D, because both angles are indicated to be right angles, and so both have the same value, $90^0$.

Clearly B(G) = B(F), because these two angles are vertical angles (or vertically opposite angles).

This demonstrates that two of the three pairs of angles put into correspondence with each other are equal, and so the third pair of angles in correspondence must also be equal:  A = E.

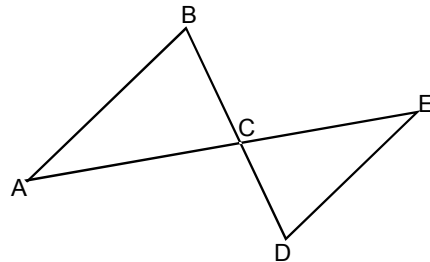Thus, property (i) of similar triangles is satisfied, and so we can conclude that

      ABC ~  EBD.


**Example:**  Suppose that lines AB and DE in the figure to the right are parallel.  Demonstrate that

      ABC ~  EDC.



**Solution:**
Propose that

            A corresponds to E
            B corresponds to D
            C corresponds to C

Then
        C = C, because the two angles are vertical
        A = E because line AE acts as a transversal crossing the parallel lines AB and ED.
                Angles A and E are opposite interior angles in that case, and so are equal.
        B = D because C = C and A = E.

Hence all three pairs of corresponding angles are equal.  Property (i) of similar triangles is satisfied by these two triangles, and so we can conclude that they are indeed similar:

      ABC ~  EDC.