UCLA Math Circle - October 21, 2007

Essential Counting Strategies

- **Sequential Counting Principle (the product rule)**
  Suppose that a procedure can be broken down into k successive tasks. If there are n_1 ways to do the first task, and n_2 ways to do the second task after the first task has been done, and so on, then there are $n_1 \times n_2 \times \cdots \times n_k$ ways to do the procedure.

- **Additive Counting Principle (the sum rule)**
  Suppose we have tasks $T_1, T_2, \ldots, T_k$ that can be done in $n_1, n_2, \ldots, n_k$ ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + \ldots + n_k$ ways to do one of these tasks.

- **Permutation**
  Permutation means arrangement of things *in a certain order*. The number of permutations of r elements taken out of a set of n elements (without repeating) is denoted $_nP_r$:

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

- **Combination**
  Combination means selection of things where *order does not matter*. The number of combinations of r elements taken out of a set of n elements is denoted $_nC_r$ or $\binom{n}{r}$:

$$_nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

- **Bijection**
  A bijection is a *one-to-one correspondence* between two sets. If there is a bijection between two finite sets, then these two sets contain exactly the same number of elements.

Examples and Exercises

1. Your friend in New York plans to take a vacation. He wants to visit you in Los Angeles and then go to Hawaii. He asks you to go with him, but you decline because you want to go to UCLA Math Circle. However, you offer to help him plan the trip and also take him to visit UCLA Math Circle. So you count the methods of transporation: from New York to LA there are 3 ways of transportation: by plane, by train, and by car. From LA to Hawaii there are 2 ways of transportation: by plane, and by boat. In how many ways can your friend go from New York to Hawaii?

2. Continuation of the previous problem. Your friend also thinks about going from New York to Hawaii via San Francisco. So instead of visiting you and UCLA Math Circle, he will visit San Francisco Math Circle. You are not happy about this option, so you give him the following choices: from New York to San Francisco, he can ride a bicycle, or walk on foot. From San Francisco to Hawaii, he can windsurf, or swim. If you count both routes (through Los Angeles or through San Francisco), in how many ways can he go from New York to Hawaii?

3. How many ways can you put 8 mutually non-attacking rooks on a standard 8×8 chessboard?

4. Explain why $\binom{n}{r} = \binom{n}{n-r}$.

5. A license plate in a certain state consists of 4 digits, not necessarily distinct, and 2 letters, also not necessarily distinct. These six characters may appear in any order, except that the two letters must appear next to each other. How many distinct license plates are possible?

6. How many rearrangements can be made of the letters in the following words: UCLA, CIRCLE, MATHEMATICAL?

7. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

8. How many four-digit positive integers have at least one digit that is a 2 or a 3?

9. n persons sit around a circular table. How many of the $n!$ arrangements are distinct, i.e., do not have the same neighboring relations?

10. Before Rick can open his gym locker, he must remember the combination. Two of the numbers of the three-term sequence are 17 and 24, but he has forgotten the third, and does not know the order of the numbers. There are 40 possibilities for the third number. At ten seconds per try, at most how long will it take him to exhaust all possibilities?

11. Three tiles are marked X and two other tiles are marked O. The five tiles are randomly arranged in a row. What is the probability that the arrangement reads $XOXOX$?

12. There are 15 distinct air conditioning vents in a movie theater. To keep the air fresh, at least one of the vents has to be on at all times. In how many ways can this be done?

13. In how many ways can TRIANGLE be spelled by moving either down or diagonally to the right in the following Figure?

$$
\begin{array}{cccccccc}
T \\
R & R \\
I & I & I \\
A & A & A & A \\
N & N & N & N & N \\
G & G & G & G & G & G \\
L & L & L & L & L & L & L \\
E & E & E & E & E & E & E & E
\end{array}
$$

14. A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

15. Six ants simultaneously stand on the six vertices of a regular octahedron, with each ant at a different vertex. Simultaneously and independently, each ant moves from its vertex to one of the four adjacent vertices, each with equal probability. What is the probability that no two ants arrive at the same vertex?

16. On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

17. Consider a 4×5 array of points, each of which is 1 unit away from its nearest neighbors. Determine the number of non-degenerate triangles (i.e., triangles with positive area) whose vertices are points in the given array.

18. Let S be the set of all points with coordinates (x, y, z), where x, y, and z are each chosen from the set $\{0, 1, 2\}$. How many equilateral triangles have all their vertices in S?

19. Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

20. How many non-similar triangles have angles whose degrees measures are distinct positive integers in arithmetic progression?

21. An envelope contains eight bills: 2 ones, 2 fives, 2 tens, and 2 twenties. Two bills are drawn at random without replacement. What is the probability that their sum is $20 or more?

22. Call a number *prime-looking* if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

23. An object in the plane moves from one lattice point to another. At each step, the object may move one unit to the right, one unit to the left, one unit up, or one unit down. If the object starts at the origin and takes a ten-step path, how many different points could be the final point?

24. A bug starts at one vertex of a cube and moves along the edges of the cube according the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

25. How many non-empty subsets of S of $\{1, 2, 3, \dots, 15\}$ have the following properties?

   (a) No two consecutive integers belong to S;
   (b) If S contains k elements, then S contains no number less than k.

26. In a 10×10 array of evenly spaced points, determine the number of squares with all their vertices belonging to the array.

27. 100 lightbulbs are arranged in a 10×10 array, evenly spaced. Each lightbulb has its own switch. All lightbulbs are initialy switched off. Define two types of operations as follows:

    - Operation A: select an array of 5×5 lightbulbs and flip all their switches;
    - Operation B: select an array of 3×3 lightbulbs and flip all their switches.

    Questions:

    (a) How many possible on-off configurations are there for this 10×10 array of lightbulbs?
    (b) Is every possible on-off configuration obtainable via a sequence of the operations A and B?