

Math Circle
Beginners Group
October 18, 2015
Solutions

Warm-up problem

1. Let n be a (positive) integer. Prove that if n^2 is odd, then n is also odd.
(*Hint:* Use a proof by contradiction.)

Suppose that n^2 is odd and n is even.

Then n can be written in the form:

$$n = 2k$$

And n^2 can be written in the form:

$$n^2 = 4k^2$$

How does this contradict our assumption?

Since $4k^2$ is always even (any number multiplied by an even number gives an even product), our original assumption that n^2 is odd and n is even has been contradicted.

What is your conclusion?

This must mean that if n^2 has to be odd, n must be odd.

Is there another way to prove this statement?

2. Divide the given numbers and find the quotient and remainder as shown below.

$$7 \div 3 = 2 \text{ R}1$$

The equation above can also be written as $7 = 2 \times 3 + 1$.

(a) $17 \div 3 =$

$$17 \div 3 = 5 \text{ R}2$$

$$17 = 5 \times 3 + 1$$

(b) $13 \div 4 =$

$$13 \div 4 = 3 \text{ R}1$$

$$13 = 3 \times 4 + 1$$

(c) $21 \div 7 =$

$$21 \div 7 = 3 \text{ R}0$$

$$21 = 3 \times 7 + 0$$

Terminating and Non-Terminating Decimals

1. Rewrite the following decimals as fractions. Do not simplify the fractions. Use the power notation for denominators.

For example, $0.07 = \frac{7}{10^2}$, $0.00103 = \frac{103}{10^5}$.

(a) 0.37
 $= \frac{37}{10^2}$

(b) 0.123
 $= \frac{123}{10^3}$

(c) 0.0123
 $= \frac{123}{10^4}$

(d) 0.000107
 $= \frac{107}{10^6}$

2. We know that fractions that have terminating decimal expansions can be written in the form $\frac{a}{2^n 5^m}$.

We will see that fractions that have other factors in the denominator do not have terminating decimal expansions.

Let $\frac{a}{17}$ be a proper fraction. Show that it cannot be converted into a terminating decimal:

- (a) We will argue by contradiction. Assume that $\frac{a}{17}$ is equivalent to a terminating decimal $0.\underline{a_1}\underline{a_2} \dots \underline{a_n}$.¹ Denote the expression after the decimal point by b , i.e. let

$$b = \underline{a_1}\underline{a_2} \dots \underline{a_n}.$$

Rewrite the assumption that $\frac{a}{17}$ can be written as a terminating decimal as follows

(insert an appropriate denominator on the right hand side). Let

¹The digits are underlined to distinguish the number $0.\underline{a_1}\underline{a_2} \dots \underline{a_n}$ written with digits a_1, a_2, \dots, a_n from the product of the numbers $a_1 a_2 \dots a_n = a_1 \cdot a_2 \cdot \dots \cdot a_n$.

$$\frac{a}{17} = \frac{b}{10^n}$$

(b) Rewrite this equality using cross-multiplication.

$$10^n \cdot a = 17 \cdot b$$

(c) Use the fact that a is not divisible by 17 to get a contradiction.

Since a is not divisible by 17 (because the fraction $\frac{a}{17}$ is in its simplest form) and 17 does not have any common factors with 10^n , this equation can never be true. Our original assumption has been contradicted.

Therefore, a fraction of the form $\frac{a}{17}$ does not have a terminating decimal expansion.

3. Prove that a fraction (written in the simplest form) with denominator 7 cannot be written as a terminating decimal. Use the same reasoning as in the previous problem.

Using the same process as above, let us assume that a fraction $\frac{s}{7}$ (written in its simplest form) has a terminating decimal expansion of the form $0.\underline{s_1} \underline{s_2} \dots \underline{s_n}$.

The expression after the decimal point can be represented by r , i.e. let

$$r = \underline{s_1} \underline{s_2} \dots \underline{s_n}.$$

If the fraction $\frac{s}{7}$ has a terminating decimal expansion, then we can write

$$\frac{s}{7} = \frac{r}{10^n}$$

Now, s is not divisible by 7 (because the fraction $\frac{s}{7}$ is in its simplest form) and 7 does not have any common factors with 10^n . So, this equation can never be true. Our original assumption has been contradicted.

Therefore, a fraction of the form $\frac{s}{7}$ does not have a terminating decimal expansion.

Using the same method, one can show that

If a fraction has prime factors other than 2 and 5 in the denominator, it cannot be converted into a terminating decimal.

Periodic Decimals

- Convert the following fractions into decimals.

(a) $\frac{2}{9}$

Using long division, we can get $2 \div 9 = 0.2222\dots$

(b) $\frac{5}{6}$

Using long division, we can get $5 \div 6 = 0.83333\dots$

- We know that the decimals above are non-terminating. What else do you notice about the decimal expansions?

The decimal expansions have repeating or recurring digits, and the expansions go on forever.

Decimals such as these are called **periodic decimals**. It means that their decimal expansions eventually repeat themselves forever. The group of digits in the repeating section is called the **period**. Sometimes, the period starts right after the decimal point (as in part (a) above), and sometimes it starts later (as in part (b) above).

For example, $\frac{1}{3} = 0.3333\dots$ is a periodic decimal. Its period is 3 and the length of the period is 1. This decimal can also be written as $0.\overline{3}$. The period is written under a horizontal line.

The following is true.

All fractions have decimal representations that eventually repeat themselves.

- Ivy thinks this statement is not true, claiming, “We already proved last week that fractions which only have prime factors 2 and 5 in the denominator have terminating decimal equivalents, so not all fractions can have eventually periodic decimal equivalents.” Is she right? Why or why not?

Ivy is not right. Even the decimal expansions of fractions that have 2 and 5 in their denominator (which we have been referring to as terminating decimals) are periodic. This is because the 0 repeats in such decimals. So, 0.5 can be written as $0.5000\dots = 0.5\overline{0}$.

Converting fractions into decimals

1. We have used long division to convert fractions into decimals. Here is another way to convert proper fractions into decimals. Let us look at $\frac{3}{7}$.

First, multiply the numerator by 10, which gives you 30, and then divide the 30 by the denominator to get the quotient and remainder.

$$30 = 4 \times 7 + 2$$

Take the remainder, multiply it by 10 to get 20, and divide it by the original denominator.

$$20 = 2 \times 7 + 6$$

The decimal expansion is composed of the quotients that we found above. So far, we have found the beginning of the decimal expansion:

$$\frac{3}{7} = 0.42\dots$$

This is not yet complete. The process should be repeated.

Show your work below and write down the decimal representation of $\frac{3}{7}$ using the quotients method:

$$60 = \underline{8} \times 7 + \underline{4}$$

$$40 = \underline{5} \times 7 + 5$$

$$50 = \underline{7} \times 7 + 1$$

$$10 = \underline{1} \times 7 + 3$$

$$30 = \underline{4} \times 7 + 2$$

$$20 = \underline{2} \times 7 + 6$$

Explain when you can stop.

One can stop when the remainder repeats itself. This is because when the remainder repeats for the first time, the following quotients and remainders also repeat themselves, and the period starts repeating.

2. Use the process above as well as long division to find the decimal expansion of $\frac{13}{15}$.
Show all your work

$$130 = 8 \times 15 + 10$$

$$100 = 6 \times 15 + 10$$

$$100 = 6 \times 15 + 10$$

			0	0.	8	6	6
1	5	1	3.	0	0	0	0
		0					
		1	3				
		0					
		1	3	0			
		1	2	0			
		1	0	0			
			9	0			
			1	0	0		
			9	0			
			1	0			

3. Use both methods to find the decimal expansion of $\frac{3}{22}$. Show all your work.

$$30 = \mathbf{1} \times 22 + 8$$

$$80 = \mathbf{3} \times 22 + 14$$

$$140 = \mathbf{6} \times 22 + 8$$

$$80 = \mathbf{3} \times 22 + 14$$

$$140 = \mathbf{6} \times 22 + 8$$

			0.	1	3	6	3	6	3
2	2		3.	0	0	0	0	0	0
			0						
			3	0					
			2	2					
				8	0				
				6	6				
				1	4	0			
				1	3	2			
					8	0			
					6	6			
					1	4	0		
					1	3	2		
						8	0		
						6	6		
						1	4		

Converting periodic decimals into fractions

1. Stephanie is thinking of a secret number (not necessarily a whole number!). If you multiply Stephanie's number by 10, you get the same result as you would if you added 6 to it. What is Stephanie's number? Give your answer both as a fraction in lowest terms and a decimal.

Let x be the number that Stephanie is thinking.

$$10x = x + 6$$

$$9x = 6$$

$$x = \frac{6}{9} = \frac{2}{3} = 0.666\dots = 0.\overline{6}$$

2. Lucy is thinking of a number as well. If you multiply Lucy's number by 1000, you get the same result as if you added 37 to it. What is Lucy's number? Give your answer both as a fraction in lowest terms and a decimal.

Let x be the number that Lucy is thinking.

$$1000x = x + 37$$

$$999x = 37$$

$$x = \frac{37}{999} = 0.037037\dots = 0.\overline{037}$$

3. Convert the decimal number $x = 0.2222\dots = 0.\overline{2}$ into a fraction:

- (a) What is $10x$ as a decimal?

$$= 2.222\dots = 2.\overline{2}$$

- (b) What is $10x - x$ as a decimal?

$$= 2.222\dots - 0.222\dots = 2.\overline{2} - 0.\overline{2} = 2$$

- (c) Since $10x - x = 9x$, what is the value of $9x$?

$$9x = 2$$

- (d) Use part (c) to find representation of x as a fraction?

$$x = \frac{2}{9}$$

4. Using the same process as above, convert the periodic decimal $y = 0.4666\dots = 0.4\overline{6}$ into a fraction.

$$100y = 46.\overline{6}$$

$$10y = 4.\overline{6}$$

$100y - 10y = 46.\overline{6} - 4.\overline{6} = 42$ (Notice that we find the difference $100y - 10y$ this time, and not $10y - y$. This makes the calculation easier because the expression after the decimal point is the same ($\overline{6}$).)

$$90y = 42$$

$$y = \frac{42}{90} = \frac{14}{30} = \frac{7}{15}$$

5. Convert $z = 0.484848\dots = 0.\overline{48}$ into a fraction.

$$100z = 48.\overline{48}$$

$$z = 0.\overline{48}$$

$100z - z = 48.\overline{48} - 0.\overline{48} = 48$ (Notice that we find the difference $100y - y$ this time. This makes the calculation easier because the expression after the decimal point is the same ($\overline{48}$).)

$$99z = 48$$

$$z = \frac{48}{99} = \frac{16}{33}$$

6. Convert $w = 0.148148148\dots = 0.\overline{148}$ into a fraction.

$1000w = 148.\overline{148}$ (The first step should be to multiply the periodic decimal by a multiple of 10 such that the expression after the decimal point is only the period.)

$$w = 0.\overline{148}$$

$$1000w - w = 148.\overline{148} - 0.\overline{148} = 148$$

$$999w = 148$$

$$w = \frac{148}{999}$$