Math Circle
Beginners Group
October 11, 2015

Warm-up problems

1. Let us practice some exponents and powers! Calculate the following:

   (a) 2^2

   (b) $2^2 \times 2^3$

   (c) $\dfrac{2^4}{2^2}$

   (d) $\dfrac{2^{2^2}}{4^4}$

   (e) $(x \cdot y)^3$

   (f) $(x^2 \cdot y)^2$

2. Simplify the following expressions.

(a) $\dfrac{3^{20} \cdot 7^6}{3^2 \cdot 7^2 \cdot 5^2 \cdot 2^2}$

(b) $\dfrac{u^4 \cdot v^3}{(u^2 \cdot v)^4}$

3. Which is bigger: 2^{3^4} or 4^{3^2}?

4. Find the value of x in $\dfrac{4}{5} = \dfrac{32}{x}$. Use cross-multiplication.

Fractions and Decimals
October 11, 2015

1. Find the least common multiples for the following sets of numbers.

    (a) 2 and 3

    (b) 4 and 16

    (c) 11 and 12

    (d) n and $n + 1$

    (e) 5, 7 and 11

    (f) n, n^2, n^3

2. One of your friends who really likes fractions has offered you a series of trades. For each of the following, decide whether or not you should take the trade (without converting the fractions to decimals). All of the fruits of the same type are identical.

    (a) He will give you $\frac{3}{10}$ of his apple if you give him $\frac{5}{14}$ of yours.

    (b) He will give you $\frac{9}{12}$ of his watermelon if you give him either $\frac{11}{15}$ or $\frac{15}{20}$ of yours.

(c) He will give you $\frac{43}{44}$ of his mango if you give him $\frac{43}{3}$ of $\frac{1}{15}$ of yours.

3. Find two fractions between $\frac{2}{3}$ and $\frac{4}{5}$.

4. Convert the following fractions into decimals and say which one is bigger.

   (a) $\frac{7}{4}$, $\frac{42}{25}$

   (b) $\frac{3333}{500}$, $\frac{667}{100}$

   (c) $\frac{9}{4}$, $\frac{226}{1000}$

How else can you compare fractions without converting them to decimals or taking a common denominator?

5. Convert the following fractions into decimals.

   (a) $\frac{1}{3}$

   (b) $\frac{11}{13}$

   (c) $\frac{1}{7}$

   (d) $\frac{25}{99}$

6. What do you think is common in the decimals expansions in problem 4?

7. What do you think is common in the decimals expansions in problem 5?

8. What do you think might be the cause of this? (Think about the denominators.)

   (a) Find the prime factorization of the following numbers:

      i. 4

      ii. 25

      iii. 100

      iv. 500

      v. 1000

   (b) Find the prime factorization of the following numbers:

      i. 3

      ii. 7

      iii. 13

      iv. 99

(c) Can you now say what is the difference between the denominators of problem 4 and problem 5?

9. To summarize, fractions with denominators that have _____ factors of only ____ and ____ have terminating decimal expansions. Fractions whose denominators have _____ factors other than ____ and ____ have decimal expansions that go on forever (non-terminating decimals).

10. Can you give a name to this property of fractions? Be creative!

11. Circle all the fractions that have terminating decimal expansions. Underline all the fractions that have non-terminating decimal expansions.

$$\frac{1}{7} \qquad \frac{3}{5} \qquad \frac{64}{125} \qquad \frac{13}{30} \qquad \frac{7}{35} \qquad \frac{19}{256} \qquad \frac{11}{160} \qquad \frac{57}{6} \qquad \frac{1}{300} \qquad \frac{37}{1200}$$

12. Can you think of a fraction whose decimal representation neither terminates nor repeats itself?

13. Convert the following decimals into fractions. What do you already know about the fractions below?

(a) 0.80

(b) 0.105

(c) 0.25

(d) 0.125

Let us prove the property that we discovered above: Fractions whose denominators only have powers of 2 and 5 in their decomposition have terminating decimal expansions.

(a) Prove that if a fraction has denominator $2^n 5^n$, i.e. it can be written in the form $\frac{a}{2^n 5^n}$, then the decimal equivalent terminates.

(b) Prove that if a fraction has denominator $2^n \cdot 5^{n+k}$, i.e. it can be written in the form $\frac{a}{2^n \cdot 5^{n+k}}$, then the decimal equivalent terminates. (Hint: Multiply the numerator and denominator by an appropriate number so that you can use part (a).)

(c) Similarly, prove that if a fraction has denominator $2^{n+k} \cdot 5^n$, then the decimal equivalent terminates.

(d) State the property that denominators of fractions with terminating decimals must have.

14. Can you prove that fractions whose denominators have prime factors other than 2 and 5 do not have terminating decimal expansions?

(a) What is an example of such a fraction?

(b) Let us assume that this fraction does have a terminating decimal expansion. What form can fractions with terminating decimal expansions be written in?

(c) Explain why the fractions you obtained above can never be equal using cross-multiplication.

15. Other than using long division, can you think of another way to convert fractions that have the property you discovered in problem 13 (d) into decimals? (Hint: Multiply the numerator and denominator by an appropriate number.) Explain this method in your words below.

16. Use the method you found above to convert the following fractions into decimals.

(a) $\dfrac{3}{4}$

(b) $\dfrac{18}{20}$

(c) $\dfrac{137}{250}$

(d) $\dfrac{11}{50}$

(e) $\frac{5}{16}$

(f) $\frac{2^{2^2}}{5^{2^2}}$

17. Prove that the decimal equivalents of the following fractions terminate.

(a) $\frac{3^{20} \cdot 7^6}{3^2 \cdot 7^2 \cdot 5^2 \cdot 2^2}$

(b) $\frac{2013 \cdot 2012}{503 \cdot 11 \cdot 61 \cdot 3 \cdot 5^4}$