

Math Circle
Beginners Group
October 4, 2015
Solutions

Warm-up problem

You have 100 bags of coins. Each bag has 100 coins, but only one of these bags has special gold coins. Each gold coin weighs 1.01 ounce and all the other mediocre coins weigh 1 ounce each. You also have a weighing scale, but it is running out of battery. You can only use the scale once. How will you identify the bag with the special gold coins?

Number the bags from 1 to 100. Take one coin from the first bag, two from the second, three from the third, so on, and a hundred coins from the 100th bag. Weigh all of these coins together. If all the coins were mediocre, the total weight would be $[1 \times (1 + 2 + 3 + \dots + 100)] = 5050$ ounces.

If the gold coins are in the first bag, the weight shown on the machine would be 0.01 ounce higher since we weighed only one coin from the first bag.

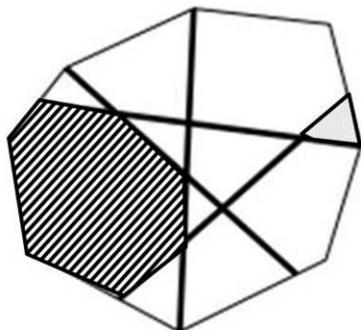
If the gold coins are in the second bag, the weight would be 0.02 ounces higher since we weighed two coins from the second bag.

Similarly, if the gold coins are in the 57th bag, the weight would be 0.57 ounces higher.

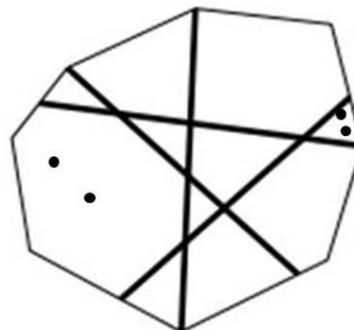
If the gold coins are in the 100th bag, the weight would be 1 ounce higher.

Depending on the total weight we see on the machine, we can find which bag the gold coins are in.

1. Four roads are crossing as shown below. Plant four trees so that the number of trees on both sides of each road is the same. Put your answer in the box, and use the other maps for experimenting.



Picture 1



Picture 2

Picture 1 shows two shaded areas. The striped area shows the region common to all roads on one side. The grey area shows the region common to all roads on the other side. Since the two areas do not coincide, we can plant two trees in each, as shown in picture 2. There can be more answers.

2. A mathematician took his 5 children out for pizza.

- Maria said, *“I must have tomatoes on my pizza, but no sausage, please!”*
- Daria said, *“No tomatoes for me!”*
- Nicolas continued, *“I like tomatoes, but can’t stand mushrooms”.*
- Igor said, *“I don’t like mushrooms too, but I would like sausage”.*
- Peter said, *“I would like mushrooms”.*

It is clear that you cannot order just one pizza to satisfy all tastes. Is it possible to order two pizzas so that everyone is happy?

Let us think about the options we would have if we wish to order two pizzas:

Tomato/sausage and mushroom

Mushroom/sausage and tomato

Tomato/mushroom and sausage

We cannot order the first set of pizzas because Maria’s taste would not be satisfied.

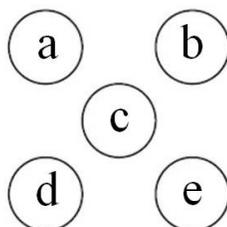
We cannot order the second set of pizzas because Igor’s preferences would be ignored.

We cannot order the third set of pizzas because Nicolas’s choice would not be satisfied.

3. Put a non-zero digit inside of each of the circles in such a way that

- the sum of the digits in the two circles in the top row is 7 times smaller than the sum of the rest of the digits; and
- the sum of the digits in the two circles on the left is 5 times smaller than the sum of the rest of the digits.

Then show that the problem has a unique solution.



- (a) Let S be the sum of all the digits in the circles. What portion of S is the sum of the two digits in the top row?

$$a + b + c + d + e = S$$

$$\text{And } a + b = \frac{1}{7}(c + d + e)$$

$$\text{So, } a + b = \frac{1}{8}S$$

Therefore, the sum of the two digits in the top row is $\frac{1}{8}$ the sum of all the digits.

- (b) What portion of S is the sum of the two digits on the left?

Similarly, the sum of the two digits on the left is $\frac{1}{6}$ the sum of all the digits.

- (c) Using (a) and (b), find what numbers should S be divisible by.

Since $\frac{1}{8}$ and $\frac{1}{6}$ of S are natural numbers, S must be divisible by 8 and 6.

- (d) List the possible values of S . Which of these does not work, and why?

The possible values of S are 24, 48, and so on. 48 and numbers greater than 48 do not work because the highest number that five non-zero digits can add up to is 45. Therefore, the sum of all the digits is 24.

- (e) Find the digits in the circles using the value of S .

The sum of the digits in the top row is $\frac{1}{8} \cdot 24 = 3$. So, we can have $a = 1$ and $b = 2$ OR $a = 2$ and $b = 1$.

The sum of the digits in the left row is $\frac{1}{6} \cdot 24 = 4$. So, we can have $a = 1$ and $d = 3$ OR $a = 2$ and $d = 2$ OR $a = 3$ and $d = 1$.

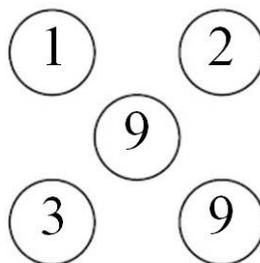
Now, we must eliminate from the options that we have. Students should try and check what each option implies. We can already see that a cannot be equal to 3.

If $a = 2$ and $b = 1$ and $d = 2$, the sum of the other two digits must equal 19, which is not possible.

If $a = 1$ and $b = 2$ and $d = 3$, the sum of the other two digits must equal 18, which works.

Therefore, c and e can only be equal to 9.

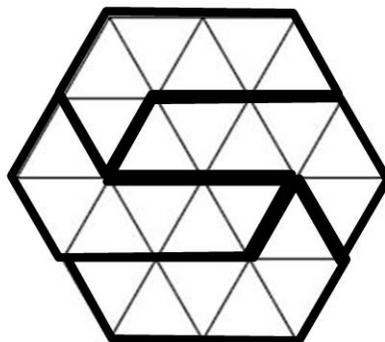
Therefore, the digits are $a = 1$, $b = 2$, $c = 9$, $d = 9$ and $e = 9$.



(f) Why does this show that there is only one solution?

Since we eliminated all possible answers, this is the only solution.

4. Cut the hexagon below into 4 parts equal in shape and size. You can cut only along the grid shown on the pictures.



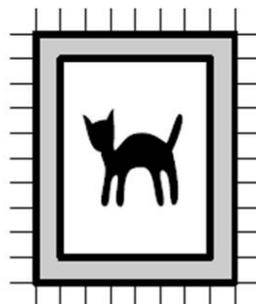
We know that each part should have 6 triangles since $24 \div 4 = 6$.

5. A monkey becomes happy if it eats 3 different types of fruit. You have 20 pears, 30 bananas, 40 apples and 50 mandarins. How many monkeys can you make happy?

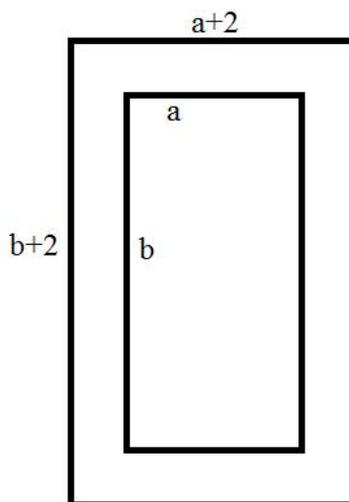
In all, there are 140 fruits. $140 \div 3 = 46.67$. Therefore, at most, we can make 46.67 monkeys happy. Since that is not possible, the natural number closest to 46.67 that is a multiple of 3 is 45. We should try to split the fruits among 45 monkeys such that each monkey gets 3 different fruits. This is how the fruits will be split.

| | |
|--------------|----------------------|
| 20 monkeys | 25 monkeys |
| 20 mandarins | 25 mandarins |
| 20 apples | 20 apples +5 bananas |
| 20 bananas | 5 bananas+20 pears |

6. George drew a rectangle on a square paper and then drew a picture inside of the rectangle. After that, he surrounded the picture by a frame which was one square wide, as shown below. It turned out that the area of the frame was equal to the area of the rectangle inside of which the picture was drawn. What are the possible sizes that George's picture could have had? List all possibilities and prove that there are no others.

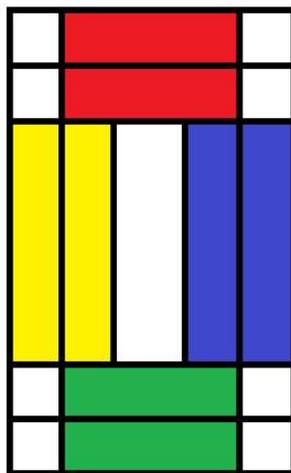


The frame is one square wide, and the dimensions of the frame are as shown below.



The area of the inside rectangle is $a \cdot b$. The area of the frame must be $2a + 2b + 4$. We know that $a \cdot b = 2a + 2b + 4$.

Since the area of the frame and the rectangle is the same, every point in the frame should fit inside the rectangle. As shown the figure below (the figure is not to scale), we have tried to fit parts of the frame into the rectangle. The red portion of the frame takes place in the rectangle directly below it. Similarly, the blue, green and yellow parts also fit into the rectangle. However, we are still left with the eight corner empty squares in the outer rectangle which need to find place within the white space inside the rectangle.



Remember that the area of the frame and that of the rectangle is the same. Therefore, whatever we are left in the frame must fit in the white space inside the smaller rectangle. The total area of what is still left to be placed is 8 since all the squares are of side 1. This means that the white space has area 8.

The only possible dimensions of a white space inside with area 8 are 1×8 and 2×4 . This means that the dimension of the rectangle inside are 3×10 and/or 4×6 .

Now, we must check if these work using the area equation above.

For 3×10 , $3 \cdot 10 = 2 \cdot 3 + 2 \cdot 10 + 4 = 30$.

And for 4×6 , $4 \cdot 6 = 2 \cdot 4 + 2 \cdot 6 + 4 = 24$.

Therefore, the sizes that George's picture could have had are 3×10 as well as 4×6 , but no other.