# Graphs and Geometry

## Los Angeles Math Circle

### 9 May 2021

# 1   Graphs

1. There are 100 cities in a country, and each road connects two of those cities. If each city has 4 roads going out of it, how many roads are there in total?

2. There are 15 cities in a country, each connected to at least 7 other cities by roads. Show that it is possible to drive from any city to any other city using the roads.

3. Prove that in any group of 6 people there are either 3 people who all know each other, or 3 people none of whom know each other. All acquaintances are assumed to be mutual. (Meaning that if John knows Pete, than Pete necessarily knows John)
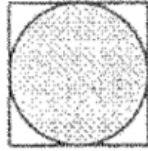
# 2　Geometry

1. Definition: For the next problem, let us introduce everyone (or remind of) a few concepts. 1. In a triangle ABC a median from a vertex to the opposing side is a segment connecting the vertex to the middle of the opposing side.

2. In the same setting, an angle bisector of an angle of a triangle is the segment from the vertex to the opposing side which divides the angle at the vertex into two equal parts. 3. Finally, an altitude from a vertex is a segment to the line containing the opposing side which makes a right angle with that line.

3. (a) Show that if $\triangle ABC$ is isosceles with $AB = BC$, then the median, and altitude from vertex $B$ to $AC$ coincide. For this problem, you are only allowed to assume triangle equality tests.

   (b) In the same setting, show that the angle bisector and the altitude coincide. Conclude that all 3 of the altitude, angle bisector and the median coincide.

4. $\triangle ABC$ is isosceles with $AB = BC$. It is known that one of the sides $AB$ and $AC$ is 7, and the other is 3. Which is which?
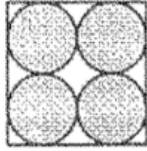
5. Let $A$, $B$, $C$ be points on a circle $\omega$. Let $P$ be a point such that the line $PB$ is tangent to $\omega$. Also let $A_1$ be the foot of the altitude from $P$ to $AB$, and $C_1$ be the foot of the altitude from $P$ to $CB$. Show that $A_1C_1 \perp AC$.

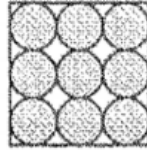# 3   Math Kangaroo Area Approximation

1. Each of the pictures below shows a square with a side of 1 and shaded circles. In which picture is the shaded area the greatest? Justify your answer.
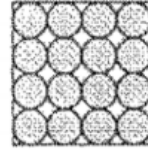


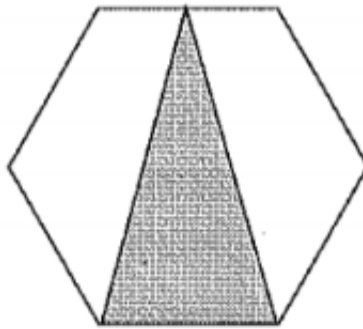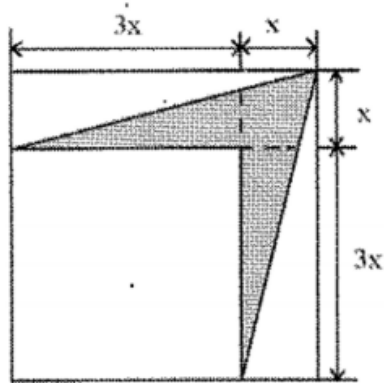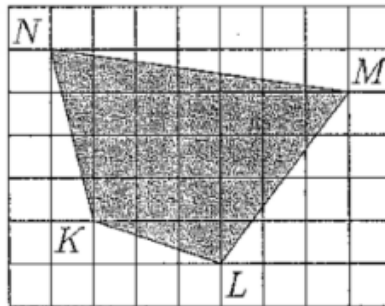Picture 1          Picture 2          Picture 3          Picture 4

2. What is the ratio of the area of the shaded triangle to the area of the regular hexagon?

3. Find the area of the shaded portion with respect to $x$.



4. The diagram shows a shaded quadrilateral $KLMN$ drawn on a grid. Each cell of the grid has sides of length 2cm. What is the area of $KLMN$?

5. The large triangle shown in the picture was divided into 36 small equilateral triangles, each with an area of 1. Find the area of triangle $ABC$.