FRACTIONS, DECIMALS, PERIODS, (IR)RATIONALS

MATH CIRCLE (BEGINNERS) 10/09/2011

(1) Here is a slightly different way to do long division and convert a fraction to a decimal. Let's try it with 3/7. First, multiply the numerator(=3) by 10, which gives you 30, then divide 30 by the denominator(=7), so you get 4, remainder 2. You can express this all on one line like so:

$$30 = 4 * 7 + 2$$

Next, take the remainder(=2), multiply *it* by 10, and divide the result again by 7. So the next line looks like this:

$$20 = 2 * 7 + 6$$

Continue on this way, multiplying each remainder by 10 and then dividing by 7 and writing it on one line. I've rewritten the first two lines below along with the third line. You do the next few lines. When should you stop? (Hint: Don't do more than 10 lines total... but you might not need to do that many...) When you're done, you will have computed the decimal expansion of 3/7 by looking at the quotients–the numbers just to the right of the equal sign. So the decimal expansion of 3/7 begins with 0.428).

$$30 = \mathbf{4} * 7 + 2$$
$$20 = \mathbf{2} * 7 + 6$$
$$60 = \mathbf{8} * 7 + 4$$

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$$40 = \mathbf{5} * 7 + 5$$
$$50 = \mathbf{7} * 7 + 1$$
$$10 = \mathbf{1} * 7 + 3$$
$$30 = \mathbf{4} * 7 + 2$$

We can stop now, because the last line is the same as the first line, so it's going to go on repeating forever like this. The decimal expansion is $3/7 = 0.\overline{428571}$.

(2) Do the process above to find the decimal expansion of 13/15.

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$$130 = \mathbf{8} * 15 + 10$$
$$100 = \mathbf{6} * 15 + 10$$
$$100 = \mathbf{6} * 15 + 10$$

The second and third lines repeat, so the decimal expansion is $13/15 = 0.8\overline{6}$.

(3) Do the process above to find the decimal expansion of 3/22.

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$$30 = \mathbf{1} * 22 + 8$$
$$80 = \mathbf{3} * 22 + 14$$
$$140 = \mathbf{6} * 22 + 8$$
$$80 = \mathbf{3} * 22 + 14$$

The second and fourth lines repeat, sot he decimal expansion is $3/22 = 0.1\overline{36}$.

(4) An infinite decimal which eventually repeats itself is called **periodic**, and the digits in the repeating section is its **period**. For example:
- the period of 1/3 = .33333333... is "3" and length of period is 1
- the period of 41/333 = .123123123... is "123" with length 3
- the period of 1/7 = .142857142857... is "142857" with length 6
- the period of 5/6 = .833333333... is "3" with length 1

(a) Jeff says, "When you use the technique above for converting a fraction to a repeating decimal, there are only 10 possible digits. Once you've used them all up, you have to repeat one of them, so the repeating section starts over. This means the length of the period of any repeating decimal you get from a fraction is at most 10." Is he right? Why or why not?
- No, he's not right. The reason is this: we get each new decimal digit as the quotient when we divide a multiple of 10 by the denominator. The process only repeats when two full lines are the same. But it's

quite possible for two different multiples of 10 to have the same quotient when divided by the denominator (if the denominator is greater than 10). If this happens, we'll get the same new digit in both cases, even though the two lines are different. (For example, if we compute $5/22$ the first two digits are 2, but $5/22 \neq 0.\overline{2}$. Try it out!)

(b) What (if anything) forces the process used in problems 1-3 to eventually repeat itself?

- There are only a limited number of possibilities for the REMAINDER after you divide by the denominator. Since the remainder completely determines the next line, once a remainder repeats, the whole process repeats.

(c) Without calculating it directly, how long could the period of a number of the form $\frac{k}{17}$ be? (k is any whole number.)

- We certainly can't have a period longer than 17 itself, since there are only 17 different remainders when you divide by 17, and once any remainder repeats, the whole process repeats.
- Actually, we can say that no period will be longer than 16. The reasoning is as follows: if the remainder 0 ever occurs, then each line from then on will just be $0 = 0*17+0$–this corresponds to a terminating decimal, which we can think of as having period 1 (for example, $1/4 = 0.250000000\ldots$). And if 0 never occurs, then there are only 16 possible remainders which may occur, so the maximal period length is 16.

(d) What is the maximum possible length of period of a number of the form $\frac{1}{n}$, in terms of n?

- The exact same reasoning as in part (c) applies here as well, so that the maximum possible length of period for $\frac{1}{n}$, is $n-1$.

(5) Check your answers to the previous problem by doing the process above to find the decimal expansion of 1/17. What is its period, and length of period?

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$$10 = \mathbf{0} * 17 + 10$$
$$100 = \mathbf{5} * 17 + 15$$
$$150 = \mathbf{8} * 17 + 14$$
$$140 = \mathbf{8} * 17 + 4$$

$$40 = \mathbf{2} * 17 + 6$$
$$60 = \mathbf{3} * 17 + 9$$
$$90 = \mathbf{5} * 17 + 5$$
$$50 = \mathbf{2} * 17 + 16$$
$$160 = \mathbf{9} * 17 + 7$$
$$70 = \mathbf{4} * 17 + 2$$
$$20 = \mathbf{1} * 17 + 3$$
$$30 = \mathbf{1} * 17 + 13$$
$$130 = \mathbf{7} * 17 + 11$$
$$110 = \mathbf{6} * 17 + 8$$
$$80 = \mathbf{4} * 17 + 12$$
$$120 = \mathbf{7} * 17 + 1$$
$$10 = \mathbf{0} * 17 + 10$$

Since the last and the first line repeat, the decimal expansion is $1/17 = 0.\overline{0588235294117647}$, which has period length 16, which we predicted was the maximum possible!

(6) The decimal number

$$0.1234567891011121314151617181920212223\ldots$$

is formed by writing down the digits of each whole number in order after the decimal point.

- Can you convert it into a fraction (ratio of whole numbers)? Why or why not?
    - No, you cannot, because it's not a periodic decimal (why not?), and we showed in the previous problem that any decimal you obtain from a fraction is periodic (because the process of problems 1-3 must repeat itself eventually).

(7) What is the 100th digit after the decimal point in the decimal representation of 1/7?
- Since $1/7 = 0.\overline{142857}$, every decimal position that's a multiple of 6 will cycle back to 7. The biggest multiple of 6 less than 100 is 96 = 16*6, then we get the 97-100th decimal digits are 1428, so in particular the 100th digit is **8**.

(8) What is the 2011th digit after the decimal point in the decimal representation of 1/7?

- Same as before, every 6th digit is a 7. The closest multiple of 6 to 2011 is 2010. (We can tell it's a multiple of 6 because it's even, and the sum of digits is 3. Or divide 2011 by 6 to get a remainder of 1; this also shows 2010 is divisible by 6.) So the 2010th digit is a 7, which means the 2011th digit is a **1**.

(9) What is the 1000th digit after the decimal point in the decimal from problem 6? ($0.12345678910111213141516\,17\ldots$)

- This is trickier, since the decimal is not periodic. Still, it has a pattern we can exploit–the numbers are written down in order. There are:
  - 9 one-digit numbers
  - 90 two-digit numbers
  - 900 three-digit numbers
- This means the 1000th digit will occur when a three-digit number is being written down. Now after we write down "99" we will have written 9 + 2*90 = 189 digits. So there are 811 more digits until the 1000th digit. These are occurring in three-digit numbers, so let's compute 811/3 = 270, remainder1. This means we can write 270 full three-digit numbers. The 270th three-digit number is... let's see, is it just 100 + 270 = 370? Well, that's not quite right because the 1st three-digit number is 100, the 2nd is 101, the 3rd is 102... we add 99 to get from 1 to 100, and from 2 to 101, etc. So the 270th three-digit number will be 270 + 99 = 369. Once we've completely written that there's one more digit to write ("remainder 1") to get to the 1000th digit, so the 1000th digit is a **3**, the one we write while writing down the number 370.

(10) A *rational number* is a number which can be written in the form $\frac{a}{b}$, where a and b are both integers, and $b \neq 0$. (*Integers* are whole numbers including negatives and zero; in other words, the integers are the numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.)

  (a) Fill in the blank: The decimal representation of a rational number is always _____periodic_____.

  (b) Is the reverse true? That is, if a number has a decimal representation which is _____periodic_____, does it mean that the number must be rational?

    - Yes! We saw from the previous handout that we can convert any periodic decimal to a fraction by writing down the repeating part

and dividing by 999...999, where the number of digits 9 is equal to the length of the period.

(c) Is 53 rational? Why or why not?
- Yes, 53 = 53/1.

(d) Is -35.33 rational? Why or why not?
- Yes, -35.31 = -3531/100.

(e) Is zero rational? Why or why not?
- Yes, 0 = 0/1.

(f) Can you write down or describe a number that's *irrational*–one which can't be written as a ratio of two integers? (Hint: There's been one already in this handout that you can use...)
- One example is the decimal number from problem 6... since it's not periodic, it can't be rational!

(11) Given that $1/7 = 0.\overline{142857}$, what is the fraction represented by $0.\overline{285714}$?
- $1/7*100 = 100/7 = 14.285714... = 14+0.\overline{285714}$, **therefore** $0.\overline{285714} = 100/7 - 14 = 100/7 - 98/7 = 2/7$.

(12) Given that $1/13 = 0.\overline{076923}$, what is the fraction represented by $0.\overline{923076}$?
- Method 1: Similar to how we proceeded in the previous problem. $1/13 * 1000 = 1000/13 = 76.923076... = 76+0.\overline{923076}$, **therefore** $0.\overline{923076} = 1000/13 - 76 = 1000/13 - 988/13 = 12/13$.
- Method 2: If we add the decimals digit-by-digit, we see that

$$0.\overline{076923}$$
$$+0.\overline{923076}$$
$$\rule{4cm}{0.4pt}$$
$$0.\overline{999999}$$

and since we know that $0.999999 = 0.\overline{9} = 1$, the decimal we wanted must be $1 - 1/13 = 12/13$.