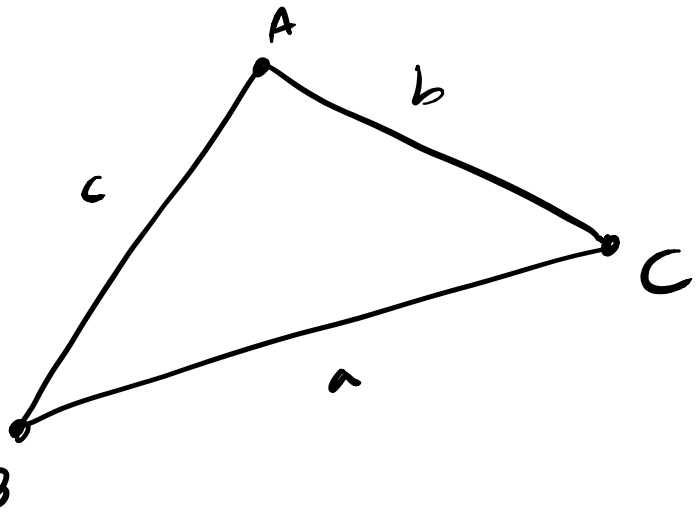
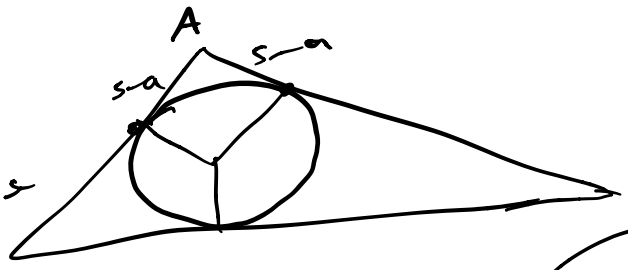


Thm. (Heron's Formula)

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{a+b+c}{2}$$



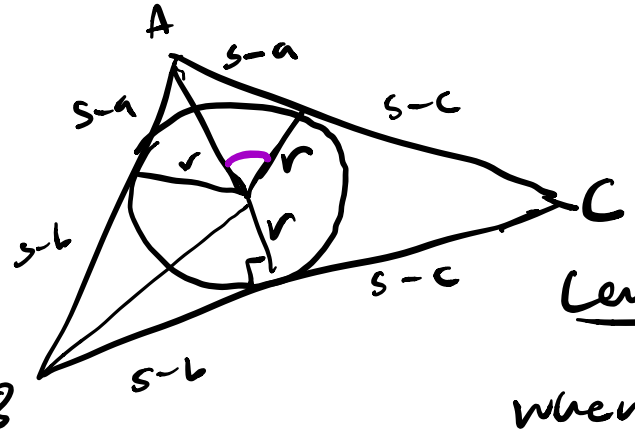
Proof 1. $[ABC] = \frac{1}{2}bc\sin A$

$a^2 = b^2 + c^2 - 2bc\cos A$

$\sin A = \sqrt{1 - \cos^2 A}$

$\cos A = \frac{b^2+c^2-a^2}{2bc}$ do algebra.

Proof 2. (EGMO ch. 5)



$\tan(90 - \frac{\angle A}{2}) = \frac{s-a}{r}$

$\tan(90 - \frac{\angle B}{2}) = \frac{s-b}{r}$

$\tan(90 - \frac{\angle C}{2}) = \frac{s-c}{r}$

Lemma. $\tan x + \tan y + \tan z = \tan x \tan y \tan z$ when $x + y + z = 180°$.

$$\frac{s}{r} = \frac{s-a}{r} + \frac{s-b}{r} + \frac{s-c}{r} = \frac{(s-a)(s-b)(s-c)}{r^3}$$

$$(r^3 \cdot s)\frac{3s - (a+b+c)}{r} \qquad \cdot (r^3 s)$$

$$2s$$

$$[ABC] = rs$$

$$r^2 s^2 = s(s-a)(s-b)(s-c) \quad \checkmark$$

$$\frac{\sin x}{\cos x} \cdot \frac{\sin y}{\cos y} + \frac{\sin z}{\cos z} = \frac{\sin(x+y)}{\cos x \cos y} + \frac{\sin z}{\cos z}$$

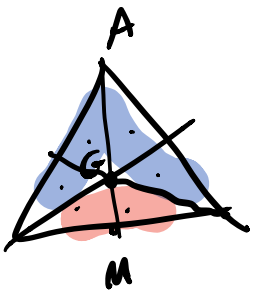
$$= \frac{\sin(x+y)\cos z + \sin z \cos x \cos y}{\cos x \cos y \cos z} \qquad -\cos(x+y)\sin z$$

$$\sin(x+y+z) = 0 \qquad (x+y+z = 180^\circ)$$

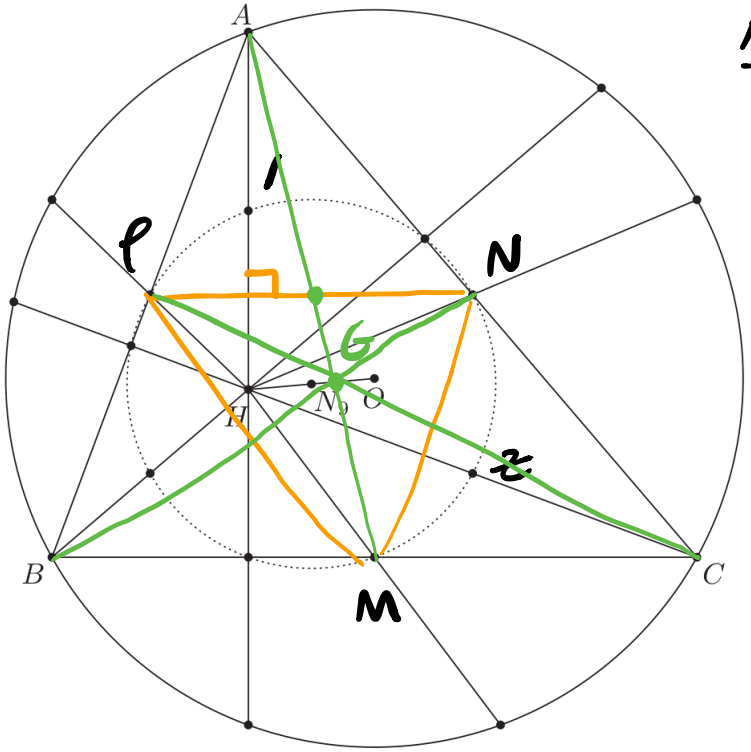
$$\cos \qquad \sin x \sin y$$

$$= \frac{\sin z(\cos x \cos y - \cos(x+y))}{\cos x \cos y \cos z}$$

$AG : GM = 2 : 1$



$\frac{AG}{GM} = \frac{\text{Blue area}}{\text{Red area}} = 2 : 1$



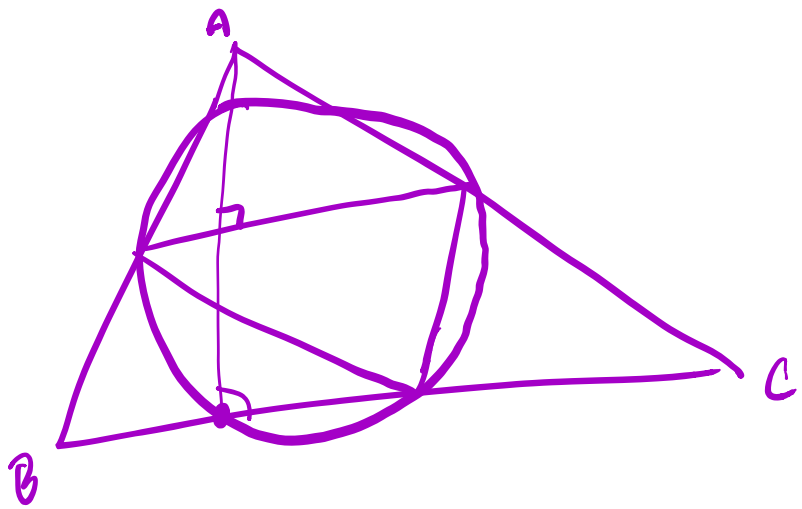
Feet of altitudes (D, E, F)

Mids of sides (M, N, P)

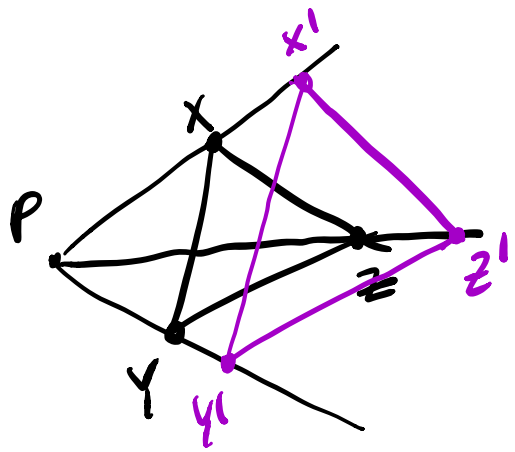
Figure 3.5C. The nine-point circle.

Mids of AH, BH, CH (X, Y, Z)

Thm. (9-pt Circle) All 9 of these points are concyclic. Their cenkr N_9 is on segment $\overline{OH}$.



Def. Homothety centered at P sends point X to point $X' \in \overleftrightarrow{PX}$ such that $\frac{PX'}{PX} = k$.



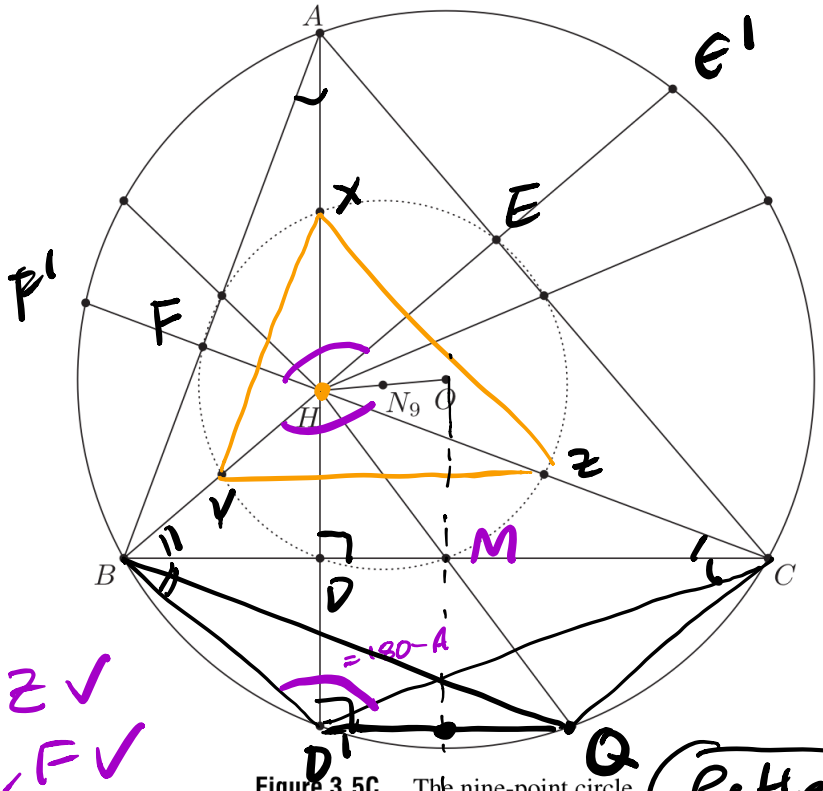


Figure 3.5C. The nine-point circle.

X = midpt(A, H)

Y = midpt(B, H)

Z = midpt(C, H)

Homothety w/ factor $1/2$ sends $A \to X$, $B \to Y$, $C \to Z$

X, Y, Z ✓

D, E, F ✓

Reflecting the orthocenter

WTS $D'D = HD$

$\triangle BHC \cong \triangle BD'C$

$\angle OCD'$
$= \angle BAD'$
$= \angle BCF$

since $\triangle BFC \sim \triangle BDA$

Q = reflection of D' over $\perp$ bis BC = $\overleftrightarrow{OM}$

$\rightarrow$ $Q \in$ circumcircle because: 1) $OD' = OQ = R$

2) $\angle BQC = \angle CD'B = 180° - \angle A$

$\rightarrow$ $HBQC$ is parallelogram since $\triangle BHC \cong \triangle BD'C \cong CQB$

$\rightarrow$ Diagonals bisect $\leftrightarrow$ parallelogram.

So M is midpt of BC and of HQ

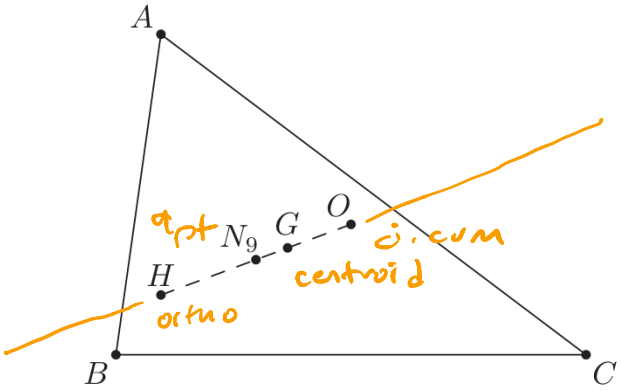
$\rightarrow$ Q is reflection of H over M.

Alternate proof

Corollary: N_a = midpt (OH)

Proof: Homothety had ratio $\frac{1}{2}$

so $\frac{HN_a}{HO} = \frac{1}{2}$ and they are collinear.



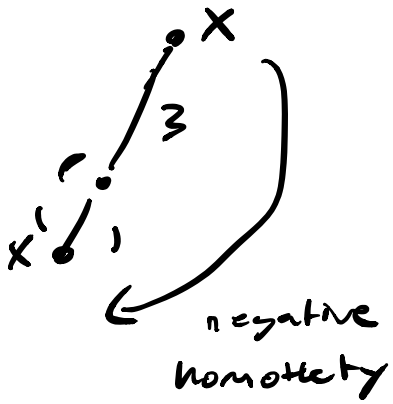


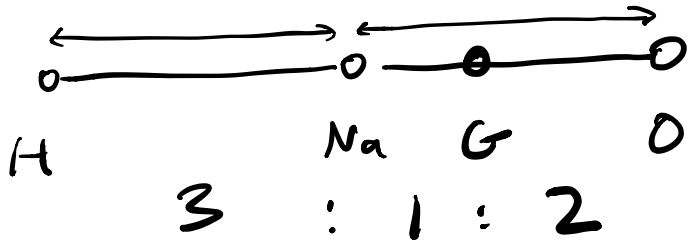
Figure 3.5D. The Euler line of a triangle.

Lemma 3.13 (Euler Line). *In triangle ABC, prove that O, G, H (with their usual meanings) are collinear and that G divides OH in a $2:1$ ratio.* **Hints**: 426 47 314

why?

Problem. Compute the ratio $HN_9 : N_9G : GO$.

$3 : 1 : 2$



Homothety around G w/ factor $-\frac{1}{2}$ sends

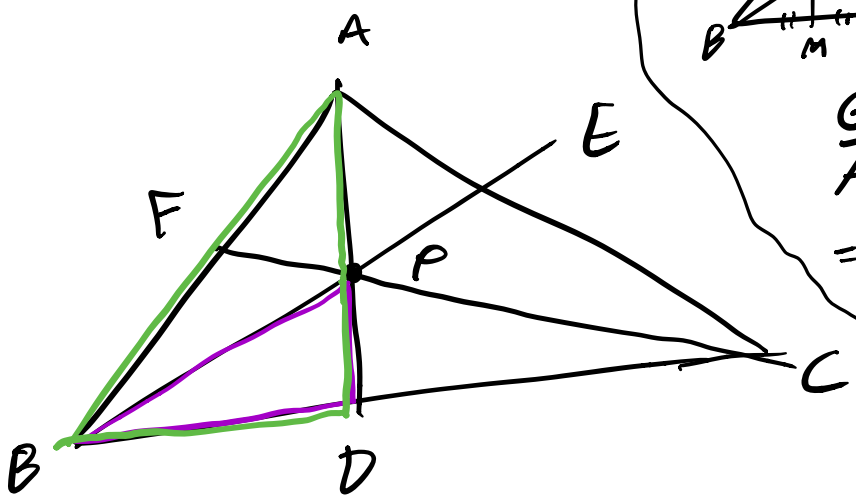
$A \to M$, $B \to N$, $C \to P$, $O \to N_9$

$\odot(ABC)$ $\odot(MNP)$

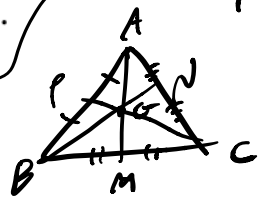
Problem 3.18. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be concurrent cevians in a triangle, meeting at P. Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

Hints: 339 16 46



Special case: centroid



$$\frac{GM}{AM} + \frac{GN}{BN} + \frac{GP}{CP} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

Hint: Proof of Ceva's theorem.

$$\frac{PD}{AD} = \frac{[PDB]}{[ADB]} = \frac{[PDC]}{[ADC]} = \frac{[PDB]+[PDC]}{[ADB]+[ADC]} = \frac{[BPC]}{[ABC]}$$

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = \frac{[BPC]+[CPA]+[APB]}{[ABC]} = \frac{[ABC]}{[ABC]} = 1.$$

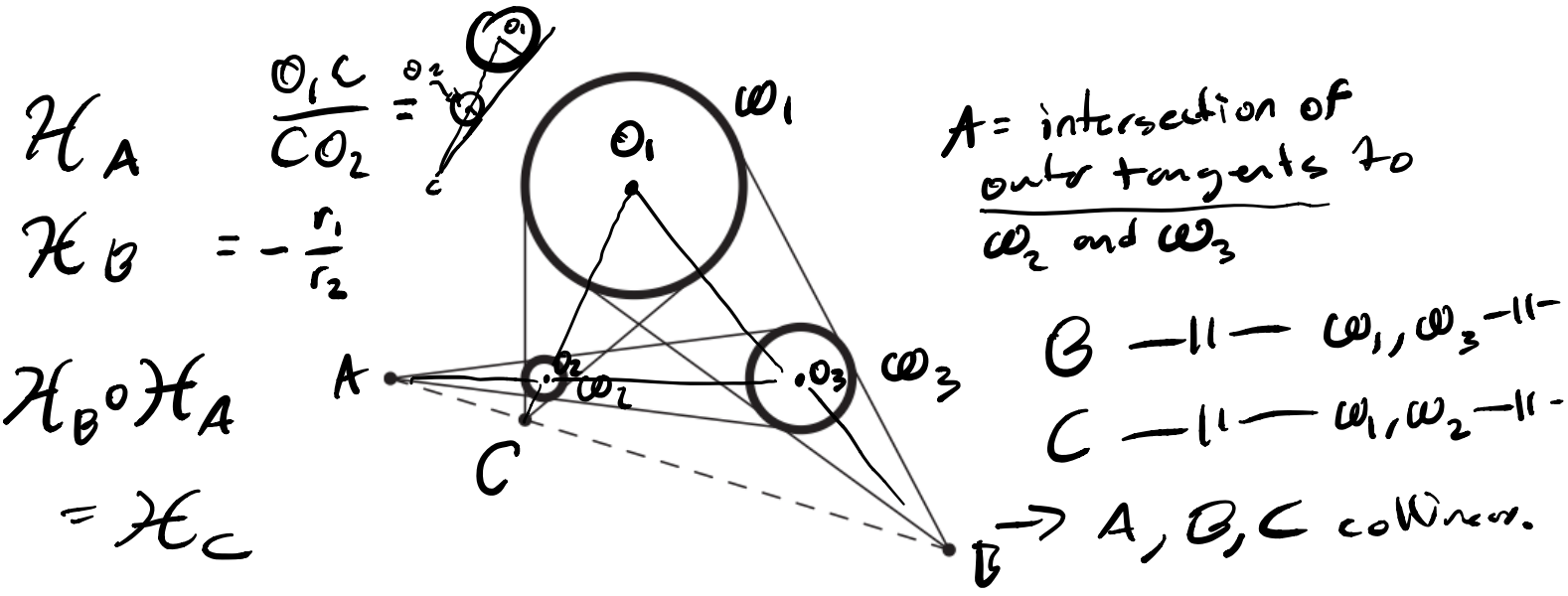
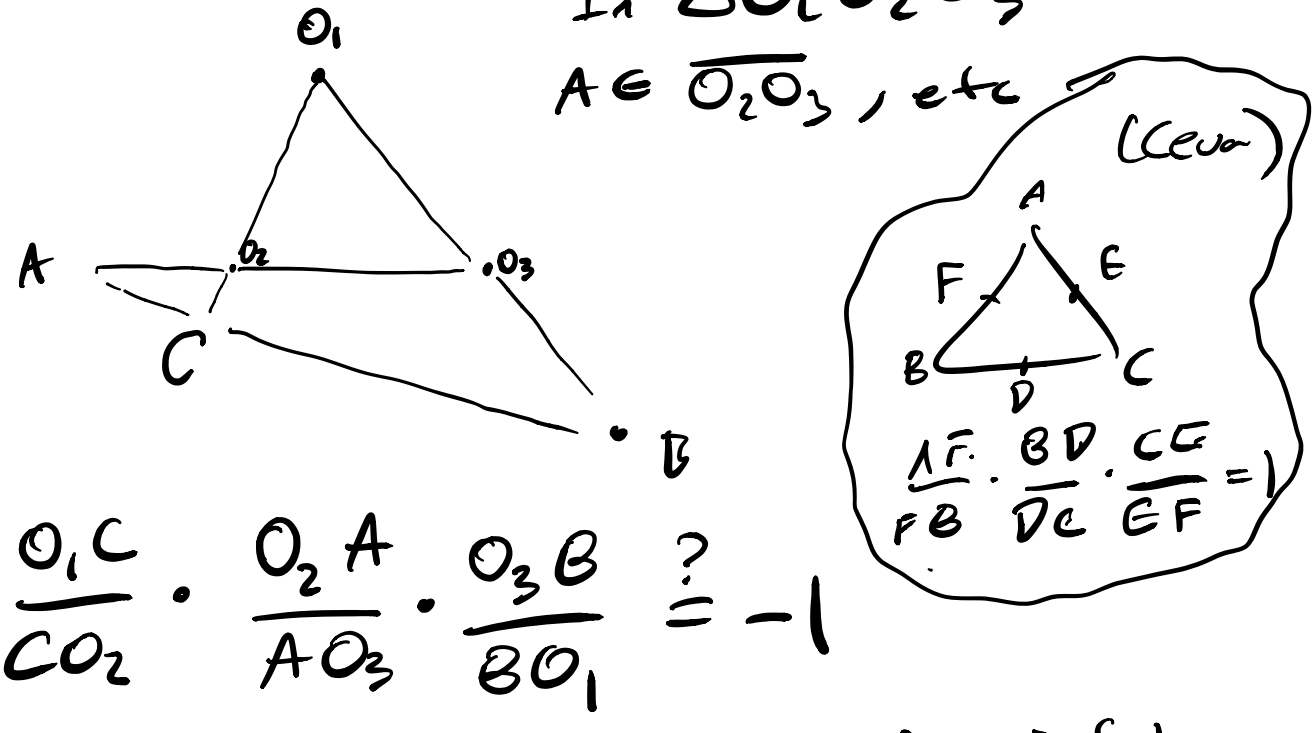


Figure 3.7A. Monge's theorem. The three points are collinear.

Hint: Menelaus' theorem.



$$= \left(\frac{-r_1}{r_2}\right)\left(\frac{-r_2}{r_3}\right)\left(-\frac{r_3}{r_1}\right) = -1 \quad \checkmark$$

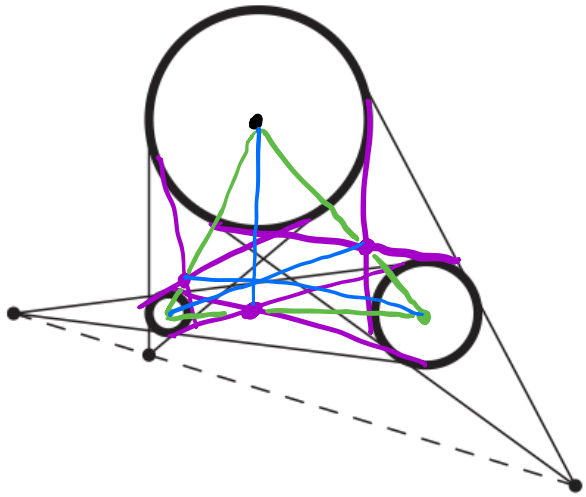
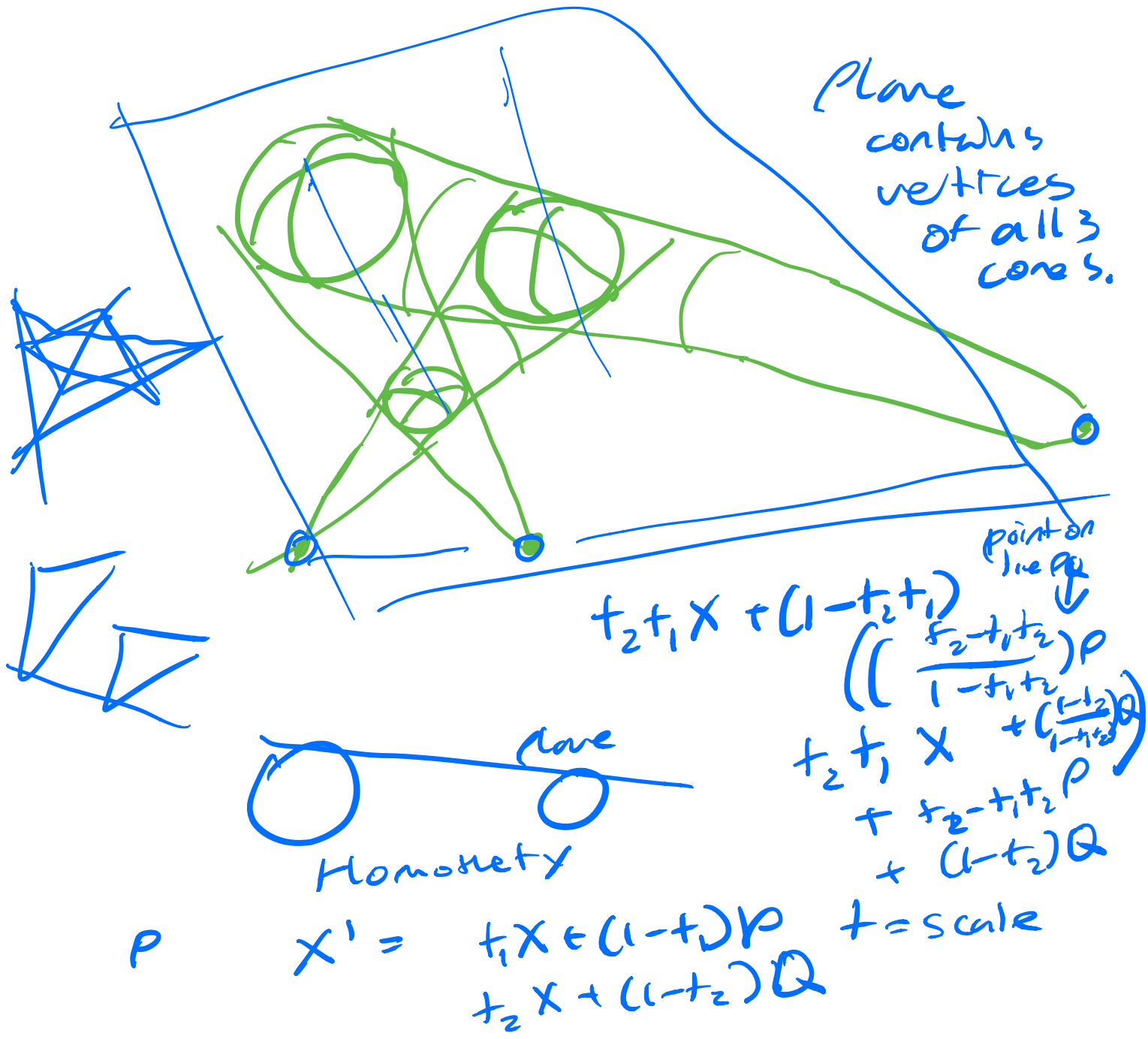



Figure 3.7A. Monge's theorem. The three points are collinear.