CRACKING THE 15 PUZZLE - PART 3 - TYING EVERYTHING TOGETHER

Review

Another Way of Calculating the Sign of a Permutation

We say that a permutation is *even* if the sign is 1 (there is an even number of inversions) and *odd* if the sign of the permutation is -1 (there is an odd number of inversions). Whether a permutation is even or odd is referred to as the *parity* of a permutation.

Problem 1. Are the following transpositions even or odd?

(1) $\sigma = \begin{pmatrix} 5 & 2 \end{pmatrix}$ on a set of 5 elements.

(2) $\sigma = \begin{pmatrix} 4 & 3 \end{pmatrix}$ on a set of 6 elements

(3) $\sigma = \begin{pmatrix} 2 & 1 \end{pmatrix}$ on a set of 3 elements.

Problem 2. Look at the permutations given on page 2 and 3. What is the correlation between the number of transpositions of a permutation and the parity of it? Can you think of an explanation as to why this correlation exists?

We will now try to prove our first theorem.

Theorem 1. *The sign of any transposition is* -1.

Before giving Theorem 1 a formal proof, let's do a proof on a concrete example.

Suppose we have a set of 10 elements

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

and we wanted to perform a transposition $\begin{pmatrix} 8 & 3 \end{pmatrix}$ on the elements.

Problem 3. How many elements are positioned between 3 and 8 (including 3 and 8)?

Problem 4. What will the set look like after applying the transposition $\begin{pmatrix} 8 & 3 \end{pmatrix}$ on the set of 10 elements?

Problem 5. How many inversions of the permutated set contain 8?

Problem 6. How many inversions in the permutated set contain 3?

Problem 7. How many inversions are there in total? Is this number even or odd? Remember that the inversion $\begin{pmatrix} 8 & 3 \end{pmatrix}$ is counted twice!

Now suppose we have a set of n elements.

$$(1 \quad 2 \quad 3 \quad \ldots \quad i \quad \ldots \quad j \quad \ldots \quad n\text{-}1 \quad n\text{-}2 \quad n)$$

and we wanted to perform a transposition $(j \quad i)$ on the elements.

Problem 8. How many elements are positioned between i and j (including i and j)?

Problem 9. What will the set look like after applying the transposition $(j \quad i)$ on the set of n elements?

Problem 10. How many inversions of the permutated set contain j?

Problem 11. How many inversions in the permutated set contain i?

Problem 12. How many inversions are there in total? Is this number even or odd? Remember that the inversion $(j \quad i)$ is counted twice!

We will now write the formal proof together as a class:

Proof. The sign of every transposition is -1.

$\square$

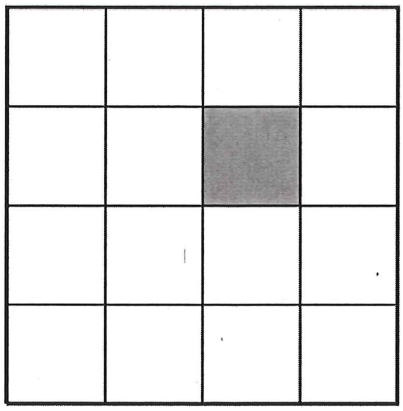
Using Theorem 1, we know that each time we apply a transposition to a set, the parity of the set switches.

Problem 13. Without doing any calculations, what is the parity of the following permutation?

$$(4\ \ 3)\circ(3\ \ 2)\circ(3\ \ 1)\circ(5\ \ 1)\circ(4\ \ 3)\circ(3\ \ 1)\circ(6\ \ 1)\circ(5\ \ 4)$$

Applying What We Have Learned to the 15 Puzzle

Problem 14. Suppose we had a blank 15 puzzle shown in the orientation below.



(1) What is the taxicab distance of the empty square to the lower-right corner?

(2) How many moves do we have to make so that the empty square is in the lower-right corner?

(3) Using the terms we have learned about permutations, how many transpositions do we have to make so that the empty square is in the lower-right corner? Explain your answer.

(4) Suppose we know that the parity of the blank 15 puzzle shown above is odd. What would be the parity of the blank 15 puzzle be once we have moved the empty square to the lower-right corner? Explain your answer.

(5) Suppose we were only told that the taxicab distance between the blank square and the lower-right corner was an odd number and that the parity of the 15 puzzle was odd. What would the parity of the 15 puzzle be once we have moved the empty square to the lower-right corner?

Problem 15. Suppose we have a 15 puzzle where the distance between the blank square and the lower-right corner is an odd number. Is the number of transpositions we need to apply to the puzzle to move the blank square to the lower-right corner odd or even?

Problem 16. Suppose we have a 15 puzzle with an odd number of inversions. How many transpositions do we need to apply to the puzzle so that there are no inversions?

While you might not realize it, you now have all the tools you need to determine whether Sam Loyd's problem with the 15 puzzle shown below has a solution or not. We will formally discuss this next class, but for now, we can make educated guesses.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Problem 17. Using your answers from Problem 15 and Problem 16, do you think a solution for Loyd's puzzle exists? Explain your answer.

Theorem 1. *The sign of any transposition is* -1.

Using this theorem, we were able to solve the last few questions in the handout. We will go over these as a class:

Problem 1. Suppose we have a 15 puzzle where the distance between the blank square and the lower-right corner is an odd number. Is the number of transpositions we need to apply to the puzzle to move the blank square to the lower-right corner odd or even?

Problem 2. Suppose we have a 15 puzzle with an odd number of inversions. How many transpositions do we need to apply to the puzzle so that there are no inversions?

Problem 3. Do you think a solution for Loyd's puzzle, as shown below exists? Explain your answer.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

Based off of our answer in Problem 3, we can now construct and prove a second theorem:

Theorem 2. *Configurations of the 15 puzzle have "similar parity" when the number of inversions and the taxicab distance from the blank space to the lower right corner are both odd or both even. Configurations of the 15 puzzle have "opposite parity" otherwise. Configurations of the 15 puzzle with opposite parities cannot be solved.*

Proof. Configurations of the 15 puzzle with opposite parity cannot be solved.

□

Are We Done? (Some Questions About Logic)

Problem 4. We know from Theorem 2 that 15 puzzles with configurations of opposite parity cannot be solved. Do you think we can now determine whether or not there's a solution for all configurations of the 15 puzzle? Why or why not?

Regardless of your answer, we need some logic to figure out what the right answer to Problem 4 is.

We use the symbol $\implies$ to mean "implies that".

If we had two statements, P and Q, $P \implies Q$ means that IF P is true, THEN Q is true.

Example. Theorem 2 can be thought of as an if-then statement, where P and Q are the following:

P : "The configuration of the 15 puzzle has opposite parity."

Q : "The configuration of the 15 puzzle cannot be solved."

This gives the following statement:

"The configuration of the 15 puzzle has opposite parity $\implies$ the configuration of the 15 puzzle cannot be solved."

Which is equivalent to

"If the configuration of the 15 puzzle has opposite parity, then the configuartion of the 15 puzzle cannot be solved."

Example. Some more if-then statements are shown below:

- It is 100 degrees outside $\implies$ It is hot outside
- Goldie is a dog $\implies$ Goldie is an animal
- It is raining $\implies$ There is traffic

Problem 5. Come up with your own if-then statement below.

The *contrapositive* of an if-then statement is when we negate each statement and turn the if-then statement around. We use the ! symbol to mean "not".

So the contrapositive of $P \implies Q$ is $!Q \implies !P$.

Problem 6. The contrapositive to the statements given in the above examples are shown below. If the statements in the examples above are true, are each of the contrapositives shown below true?

- It is not hot outside $\implies$ It is not 100 degrees outside
- Goldie is not an animal $\implies$ Goldie is not a dog
- There is no traffic $\implies$ It is not raining

Logically, all statements are equivalent to their contrapositive.

Problem 7. What is the contrapositive to Theorem 2? (Check your answer with your assistant instructor to make sure it's correct.)

The *converse* of an if-then statement is when we turn the if-then statement around without negating each statement.

So the converse of $P \implies Q$ is $Q \implies P$.

The converse of the statement is NOT always true.

Problem 8. The converse to the statements given in the above examples are shown below. If the statements in the examples above are true, are each of the converses shown below true? Give a counterexample to show how the statements below could be wrong.

- It is hot outside $\implies$ It is 100 degrees outside

- Goldie is an animal $\implies$ Goldie is a dog

- There is traffic $\implies$ It is raining

Problem 9. What is the converse to Theorem 2? (Check your answer with your assistant instructor to make sure it's correct.)

Because not all converses are true, we have to prove that the converse is true in order to use it.

Problem 10. If we knew that the configuration of the 15 puzzle had similar parity, could we conclude that the 15 puzzle was solvable? Explain your answer.

Theorem 3. *Any configuration of the 15 puzzle with similar parity is solvable.*

Because Theorem 3 is the converse of Theorem 2, we have to solve it in order to use it. Theorem 3 is not hard to prove using mathematical induction. However, we are not going to do it at the moment as induction proofs are a little bit beyond what we have learned. For now, we can take it as a fact without proving it.