CRACKING THE 15 PUZZLE - PART 2: MORE ON PERMUTATIONS AND TAXICAB GEOMETRY

Warm Up

Find the product of the following permutations by first writing the permutations in their expanded form and then calculating their products.

(1) $\begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 3 & 2 & 1 & 4 \end{pmatrix}$

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(2) $\begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 2 & 3 & 4 & 1 \end{pmatrix}$

(3) $\begin{pmatrix} 2 & 5 & 1 & 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 3 & 4 & 1 & 5 & 2 \end{pmatrix}$

Further Improving Notation

Let us take another look at the permutation

$$\sigma = \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}$$

The permutation does not shuffle the second element. Hence, writing it is redundant. Knowing that the original set consists of four elements, we can write the permutation down as

$$\sigma = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$$

This convention becomes very convenient with larger permutations.

Problem 1. In Sam Loyd's problem with the 15 puzzle, we were given a 15 puzzle with the following orientation.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

This is represented by the permutation

$$\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 15\ 14\ 16)$$

The permutation only switches the 14th and 15th elements only. What is this permutation written in the new notation?

$$\sigma = \begin{pmatrix} \quad \end{pmatrix}$$

Problem 2. The permutation $\nu = \begin{pmatrix} 3 & 1 \end{pmatrix}$ acts on a set of four elements. Write down its full version.

$$\nu = \begin{pmatrix} \quad \end{pmatrix}$$

Transpositions

A permutation that swaps two elements and doesn't shuffle anything else is called a *transposition* For example, the permutation that switches the order of the third and fifth element in a six-element set is

$$\begin{pmatrix}5 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 5 & 4 & 3 & 6\end{pmatrix}$$

Problem 3. What is the inverse of the transposition $\begin{pmatrix}5 & 3\end{pmatrix}$?

$$\begin{pmatrix}5 & 3\end{pmatrix}^{-1} = \begin{pmatrix} & \end{pmatrix}$$

Permutations as a Product of Transpositions

Any permutation can be re-written as a product of transpositions. For example, let us consider the permutation $\sigma = \begin{pmatrix}3 & 1 & 4 & 2\end{pmatrix}$.

Applying the transposition $\begin{pmatrix}3 & 1\end{pmatrix}$ to the original order of the elements gives us the following

$$\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} \xrightarrow{(3 \quad 1)} \begin{pmatrix}3 & 2 & 1 & 4\end{pmatrix}$$

Then the transposition $\begin{pmatrix}3 & 2\end{pmatrix}$ to the result gives us the following

$$\begin{pmatrix}3 & 2 & 1 & 4\end{pmatrix} \xrightarrow{(3 \quad 2)} \begin{pmatrix}3 & 1 & 2 & 4\end{pmatrix}$$

Finally, applying the transposition $\begin{pmatrix}4 & 3\end{pmatrix}$ to the result gives us the permutation $\sigma = \begin{pmatrix}3 & 1 & 4 & 2\end{pmatrix}$

$$\begin{pmatrix}3 & 1 & 2 & 4\end{pmatrix} \xrightarrow{(4 \quad 3)} \begin{pmatrix}3 & 1 & 4 & 2\end{pmatrix}$$

Instead of applying $\begin{pmatrix}3 & 1 & 4 & 2\end{pmatrix}$, we can apply $\begin{pmatrix}3 & 1\end{pmatrix}$, $\begin{pmatrix}3 & 2\end{pmatrix}$ and then $\begin{pmatrix}4 & 2\end{pmatrix}$.

So we can say that

$$\begin{pmatrix}3 & 1 & 4 & 2\end{pmatrix} = \begin{pmatrix}4 & 3\end{pmatrix} \circ \begin{pmatrix}3 & 2\end{pmatrix} \circ \begin{pmatrix}3 & 1\end{pmatrix}$$

Problem 4. Write the product of the following transpositions for a 5-element set. Remember to apply the permutation on the right first!

(1) $(5\ \ 4) \circ (3\ \ 1) \circ (3\ \ 2)$

(2) $(2\ \ 1) \circ (4\ \ 2) \circ (5\ \ 3)$

(3) $(3\ \ 2) \circ (3\ \ 2) \circ (4\ \ 1) \circ (4\ \ 1) \circ (5\ \ 2) \circ (5\ \ 2) \circ (3\ \ 1) \circ (3\ \ 1) \circ (4\ \ 3) \circ (4\ \ 3)$
Hint: You can simplify this. Look at Problem 3.

Problem 5. Re-write the permutation $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$ as a product of transpositions.

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \xrightarrow{\quad (\qquad\qquad) \quad} \begin{pmatrix} 2 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 \end{pmatrix} \xrightarrow{\quad (\qquad\qquad) \quad} \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

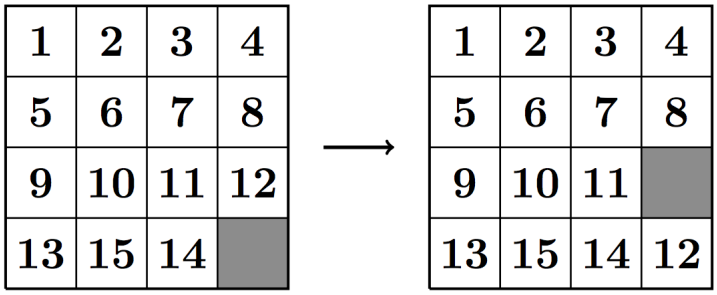
So $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix} = (\qquad\qquad) \circ (\qquad\qquad)$

Problem 6. Re-write the permutation $\begin{pmatrix} 4 & 3 & 1 & 2 \end{pmatrix}$ as a product of transpositions.

Problem 7. Re-write the permutation $\begin{pmatrix} 3 & 4 & 1 & 2 \end{pmatrix}$ as a product of transpositions.

Problem 8. Re-write the permutation $\begin{pmatrix} 3 & 5 & 1 & 2 & 4 \end{pmatrix}$ as a product of transpositions.

Problem 9. We will now see how permutations can represent the moves made on the 15 puzzle.



(1) Write down the permutation σ_1 that corresponds to the orientation of the puzzle on the left. Remember, we treat the empty square as the 16th tile!

(2) Write down the permutation σ_2 that corresponds to the orientation of the puzzle on the right as a single permutation.

(3) Write down the permutation μ that corresponds to the move shown above of the 15 puzzle.

$$\mu = \Big(\qquad\qquad \Big)$$

(4) Apply μ to σ_1. That is, calculate $\mu \circ \sigma_1$.

$$\mu \circ \sigma_1 =$$

(5) Notice that $\mu \circ \sigma_1 = \sigma_2$. Why does this make sense?

Parity of a Permutation

An *inversion* of a permutation occurs when a smaller number is moved to the right of a larger number.

For example, the permutation $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$ moves 5 to the first position so $\begin{pmatrix} 5 & 1 \end{pmatrix}$, $\begin{pmatrix} 5 & 2 \end{pmatrix}$, $\begin{pmatrix} 5 & 3 \end{pmatrix}$, $\begin{pmatrix} 5 & 4 \end{pmatrix}$ are all inversions of σ.

Note that although the words "inverse" and "inversion" are very similar, the inverse of a permutation and the inversions of a permutation are very different!

Problem 10. Write down all other inversions of the permutation $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$.

Problem 11. Write down all inversions of the permutation $\sigma = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$.

The *sign* of a permutation is defined according to the following formula:

$$sgn(\sigma) = (-1)^{N(\sigma)}$$

where $N(\sigma)$ is the number of versions of the permutation σ.

For example, the total number of inversions in $\sigma = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ is 2, so $sgn(\sigma) = (-1)^2 = 1$.

Problem 12. What is the sign of the trivial permutation?

$$sgn(e) =$$

Problem 13. Find the signs of the following permutations

(1) $sgn\begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$

(2) $sgn\begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}$

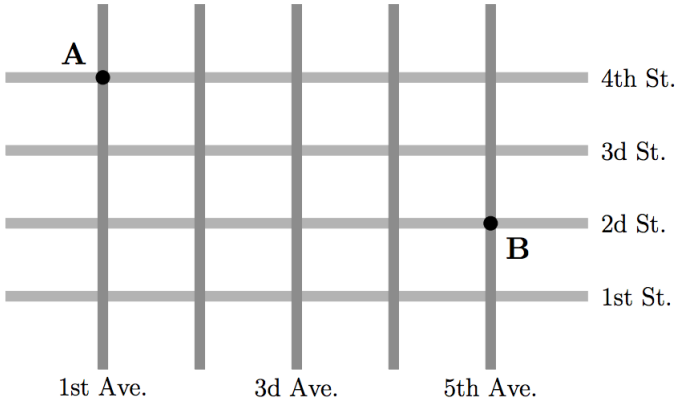
Problem 14. What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered the 16th tile.)

1	2	3	4
5	6	7	8
9	10		11
13	15	14	12

Taxicab Geometry

We are not pretty close to figuring out whether or not a solution exists for an orientation of the 15 puzzle, but we still need to learn some taxicab geometry before tying everything together.

Imagine that you take a taxicab to get from point A to point B in a city with streets and avenues forming a rectangular pattern.



Similar to Euclidean geometry, there exists a shortest path. Unlike Euclidean geometry, the shortest path is not unique.

Problem 15. On the picture above, draw two different shortest paths from A to B.

The point A lies at the intersection of the 1st Ave. and the 4th St. Let us write this fact down as follows

$$A = (1, 4)$$

B lies at the intersection of the 5th Ave. and the 2nd St.

$$B = (5, 2)$$

Let a be the distance between two neighboring avenues and let s be the distance between two neighboring streets. No matter what shortest path the cab driver chooses, he needs to drive 4 blocks East and 2 blocks South. We can write this as

$$d_{tc}(A, B) = 4a + 2s$$

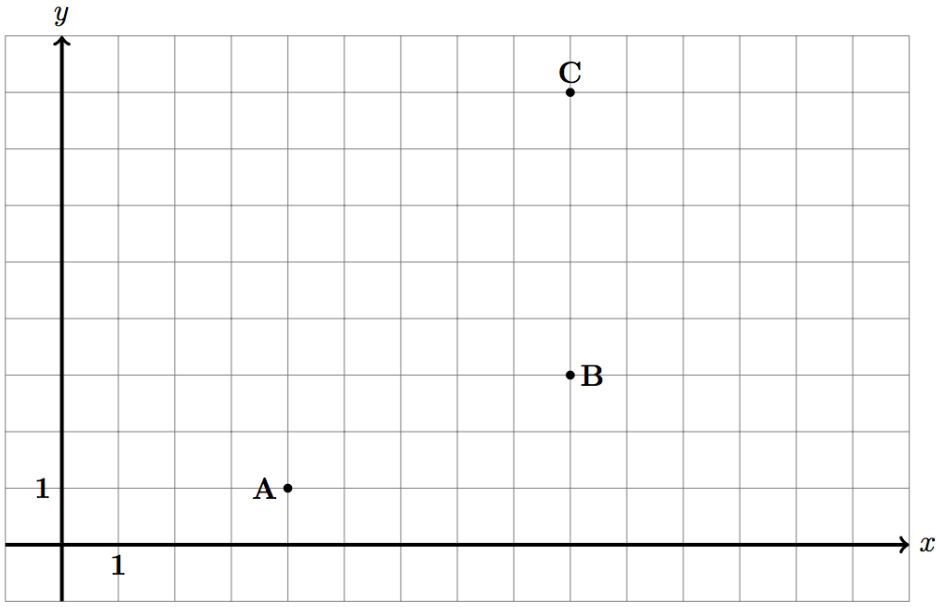
Problem 16. Find the Euclidean distance $d_E(A, B)$ between the points A and B.

$$d_E(A, B) =$$

Problem 17. Without doing any computations, put the correct sign, $\geq$, $\leq$, or $=$ between the distance below. Explain your choice.

$$d_E(A, B) \qquad\qquad d_{tc}(A, B)$$

Problem 18. For the grid below, $a = s = 1$. Find the following taxicab distances.

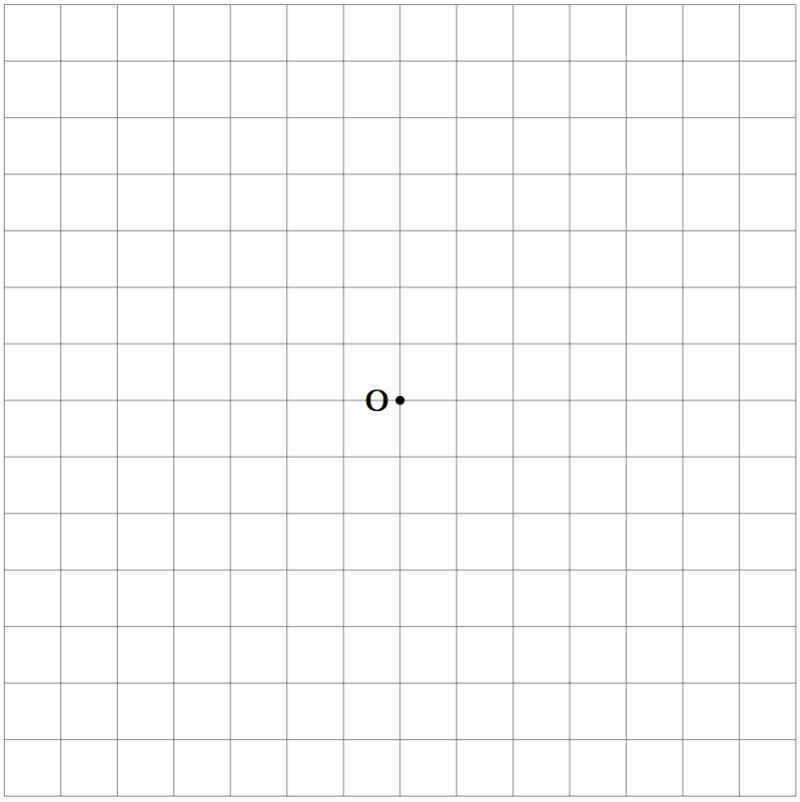


$d_{tc}(A, B) =$

$d_{tc}(A, C) =$

$d_{tc}(B, C) =$

Problem 19. On the grid below, mark all the points that have the taxicab distance 6 from the point O using a pencil. Next, mark all the points that have a Euclidean distance 6 from the point O using a pen.



Problem 20. Find the taxicab distance from the current position of the empty square to the lower-right corner of the 15 puzzle.

1	2	3	4
5	6		8
9	10	7	11
13	15	14	12

Write the following composition of transpositions as a single permutation on a set of five elements.

(1) $\sigma = (5\ \ 3) \circ (5\ \ 2)$

(2) $\sigma = (5\ \ 2) \circ (3\ \ 2) \circ (3\ \ 1)$

(3) $\sigma = (4\ \ 3) \circ (4\ \ 2) \circ (3\ \ 1) \circ (5\ \ 4)$

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Write the following permutations as a composition of transpositions. Are there an odd or even number of transpositions?

(1) $\sigma = \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}$

(2) $\sigma = \begin{pmatrix} 2 & 1 & 5 & 3 & 4 \end{pmatrix}$

(3) $\sigma = \begin{pmatrix} 3 & 5 & 2 & 1 & 4 \end{pmatrix}$

For the following permutations, list the inversions in the permutations and calculate the sign of the permutations.

(1) $\sigma = \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}$

(2) $\sigma = \begin{pmatrix} 2 & 1 & 5 & 3 & 4 \end{pmatrix}$

(3) $\sigma = \begin{pmatrix} 3 & 5 & 2 & 1 & 4 \end{pmatrix}$

Another Way of Calculating the Sign of a Permutation

We say that a permutation is *even* if the sign is 1 (there is an even number of inversions) and *odd* if the sign of the permutation is -1 (there is an odd number of inversions). Whether a permutation is even or odd is referred to as the *parity* of a permutation.

Problem 1. Are the following transpositions even or odd?

(1) $\sigma = (5 \quad 2)$ on a set of 5 elements.

(2) $\sigma = (4 \quad 3)$ on a set of 6 elements

(3) $\sigma = (2 \quad 1)$ on a set of 3 elements.

Problem 2. Look at the permutations given on page 2 and 3. What is the correlation between the number of transpositions of a permutation and the parity of it? Can you think of an explanation as to why this correlation exists?

We will now try to prove our first theorem.

Theorem 1. *The sign of any transposition is* -1.

Before giving Theorem 1 a formal proof, let's do a proof on a concrete example.

Suppose we have a set of 10 elements

$$(1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10)$$

and we wanted to perform a transposition $(8 \quad 3)$ on the elements.

Problem 3. How many elements are positioned between 3 and 8 (including 3 and 8)?

Problem 4. What will the set look like after applying the transposition $(8 \quad 3)$ on the set of 10 elements?

Problem 5. How many inversions of the permutated set contain 8?

Problem 6. How many inversions in the permutated set contain 3?

Problem 7. How many inversions are there in total? Is this number even or odd? Remember that the inversion $(8 \quad 3)$ is counted twice!

Math Kangaroo

(1) The five-digit natural number $\boxed{2}\boxed{4}\boxed{x}\boxed{8}\boxed{y}$ is divisible by 4, 5, and 9. What is the sum of the digits x and y?

(2) Alex lights a candle every ten minutes. Each candle burns for 40 minutes and then goes out. How many candles are alight 55 mintues after Alex lit the first candle?

(3) A sequence starts with 1, -1, -1, 1, -1. After the fifth term, every term is equal to the product of the two preceding terms. For example, the sixth term is equal to the product of the fourth term and the fifth term. What is the sum of the first 2013 terms?