INTRODUCTION TO HOMOTOPY THEORY

Sanath Devalapurkar

Homotopy theory is a very interesting branch of a subfield of mathematics called *algebraic topology*. Today, I'll give an introduction to a basic notion in homotopy theory, namely the notion of *homotopy groups*. My hope is that you develop an intuition for these objects, so, except for Exercise 1.7, don't think rigorously!

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§1. Homotopy. The basic objects of study in (algebraic, differential, "add a prefix here") topology is the notion of a *topological space*. You should think of a topological as a set X with "open sets", or simply, as a drawing board. For example, let $\mathbf{R}$ denote 1-dimensional Euclidean space (or the *real line*, as it is classically known). This is a topological space with "open intervals" as you know it from calculus. (Choosing these open sets leads to what is known as the *Euclidean topology*.)

The key point is that you have to include the information about *which subsets* of X are considered to be open. One other key point is that in a topological space you can draw continuous paths between points. So, other examples include the circle and the sphere, where you can draw maps between two chosen points which are on the "boundary".

Exercise 1.1. It's obvious what a path should be on a sphere, but it seems nontrivial to understand what it means on a circle. Understand the notion of a *path* on a circle. Draw out an example.

What is a map of sets? If A and B are sets, then a map from A to B, denoted $A \to B$, is a rule which specifies what the image of an element of A is in B. For example, if $\mathbf{Z}$ denotes the integers, and $\{0, 1\}$ is the set containing only the integers 0 and 1, there's a natural map $\mathbf{Z} \to \{0, 1\}$ which takes an integer to 0 if it's even and to 1 if it's odd.

Now suppose you have two topological spaces X and Y. What is a "map of topological spaces" from X to Y? Well, you cannot simply say that it's a map of the underlying sets, because, remember, to specify a topological space you need to specify which subsets are open. So a "continuous" map $X \to Y$ (aka a map of topological spaces), as in calculus, describes the image of a point $x \in X$ in the topological space Y and also says that if you've got an open set U in Y, then the preimage of all the elements of U forms an open set in X.

This can be better demonstrated via an example: any polynomial $f : \mathbf{R} \to \mathbf{R}$ is a continuous map of topological spaces. The idea is that the graph of the function is continuous, i.e., doesn't have any breaks. (This is where the word "continuous map" stems from.) The key point is that *you should think of a continuous map of topological spaces $X \to Y$ as the "image" of X in Y*. This will be manifest in the definition of a path below.

What is a good notion of *equivalence* of spaces?

We say that two spaces are *homotopy equivalent* if we can bend, stretch, etc., and *squish* them to make them "equivalent". For example, let $\mathbf{R}^n$ denote n-dimensional Euclidean space. Then we can squish this to a point, so it is homotopy equivalent to a point. Thus we say that $\mathbf{R}^n$ is *contractible*.

Exercise 1.2. A common joke in mathematics is that a topologist is someone who can't distinguish between a donut and a mug. Can you see why? (Think of homotopy equivalences.)[1]

Exercise 1.3. Can you find five letters (capital/uppercase) which are contractible?

Exercise 1.4. Give an example of two uppercase letters which can*not* be deformed into one another *without* squishing, but *can* with squishing.

Can we be more (somewhat) precise about homotopy equivalences? Let X and Y be topological spaces, and let $f_0 : X \to Y$ and $f_1 : X \to Y$ be continuous maps of topological spaces. Say that f_0 and f_1 are *homotopic* if for each real number t between 0 and 1 (inclusive), there are maps $f_t : X \to Y$ such that you can "infinitesimally deform" f_0 to each f_1 by shifting f_0 along the collection of the specified maps f_t. This simply means that we can "deform" f_0 into f_1, continuously, or without jumping.

Definition 1.5. Let X, Y be topological spaces. If there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps on X and Y, respectively, then we say that X is *homotopy equivalent* to Y.

Two homotopy equivalent spaces have the "same number of holes". Let's try finding an example.

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[1]Technically, these spaces are *homeomorphic*, which is a stronger notion than homotopy equivalence. Finding an example of two things which are homotopy equivalent but not homeomorphic is Exercise 1.4.

Exercise 1.6. Find a homotopy equivalence between the cylinder, which is a "product" $S^1 \times [0,1]$, and the circle S^1.

The next exercise will ask you to *explicitly* write down a homotopy equivalence.

Exercise 1.7. Let X be the unit circle S^1 in the complex plane $\mathbf{C}$, and let Y be the real plane with the origin removed. Define an *explicit* homotopy equivalence between S^1 and $\mathbf{R}^2 - \{(0,0)\}$. **Hint:** Use the "inclusion' $S^1 \to \mathbf{R}^2 - \{(0,0)\}$ and Pythagoras' theorem - what's a good way to assign, to a pair of nonzero real numbers, a point on the unit circle?

What this requires is for you to specify two maps $f : S^1 \to \mathbf{R}^2 - \{(0,0)\}$ and $g : \mathbf{R}^2 - \{(0,0)\} \to S^1$.

§2. Homotopies. Now, suppose $X = [0, 1]$, and let Y be a topological space. A *path* in Y is a continuous function $f : [0, 1] \to Y$; so homotopies between maps of topological spaces give homotopies between paths. A *loop* in Y is a path $f : [0, 1] \to Y$ such that $f(0) = f(1)$; this point $x = f(0)$ is called the *basepoint* of the loop.

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§3. The fundamental group. If you've got a collection of loops on a topological space, you can consider the set of loops *modulo homotopy*, which simply means that you identify two loops together if they're homotopic.

Let X be a topological space. Choose a point $x \in X$, and consider all the loops whose basepoint is x modulo homotopy. This is called the *fundamental group* of X based at x, and is denoted $\pi_1(X, x)$. If X is connected, i.e., cannot be represented as the union of two or more disjoint nonempty open subsets, then we don't even need to specify the basepoint $x \in X$. Why is this true?

Now, why not call it the fundamental **set** of X based at x? Because it looks ridiculously similar to the integers (and the rationals, reals, and complex numbers), in that it satisfies the following conditions. Suppose γ and Γ are loops based at x. Define the "product" of γ and Γ to be the composition $\gamma \circ \Gamma$.

(1) "Multiplication" of two loops (see above) is itself a loop.
(2) The multiplication is associative.
(3) There is an identity loop such that when composed with any other loop it yields the loop itself.
(4) For every loop there is an inverse loop such that when the two are composed they yield the identity.

Exercise 3.1. Draw out some pictures to convince yourself that these are satisfied. (This visualization of the "multiplication" in $\pi_1(X, x)$ may be helpful (the image has been taken from the *Princeton Companion to Mathematics*):
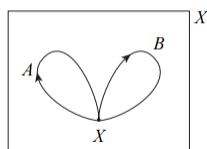
If X and Y are homotopy equivalent, then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$; in other words, there is a map $\phi : \pi_1(X) \to \pi_1(Y)$ which is a bijection which takes the multiplication $\phi(f \circ g) \to \phi(f) \circ \phi(g)$. Let's see this with the example we established below.

Exercise 3.2. Recall that $\mathbf{R}^2 - \{(0,0)\}$ is homotopy equivalent to the circle. Compute the fundamental groups of $\mathbf{R}^2 - \{(0,0)\}$ and S^1. (These are connected (why?), so forget about basepoint issues.) **Hint 1:** this is actually not that hard; just draw stuff out! **Hint 2:** Because $\mathbf{R}^2 - \{(0,0)\}$ is homotopy equivalent to the circle, it suffices to only compute fundamental group of *one* of the two spaces. It's easier to compute π_1 of S^1.

In topology, $S^1 \times S^1$ called a torus; why? A torus is simply a donut. Let's imagine a circle; choose a point on this circle. Then draw another circle, "perpendicular" to the first circle, around this point. If you push the new circle around the old circle, until you reach the starting point, you end up with a donut. Visually[2], the first step is the following, namely choosing a point on a circle (which is tilted to represent "perpendicularity"):
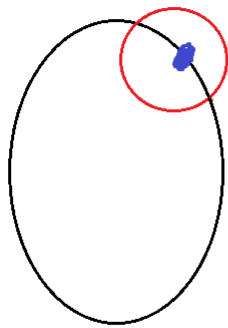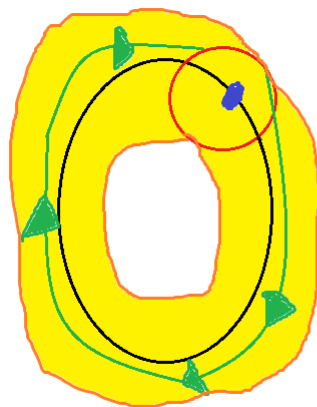

The second step is pushing the new circle around the old circle:
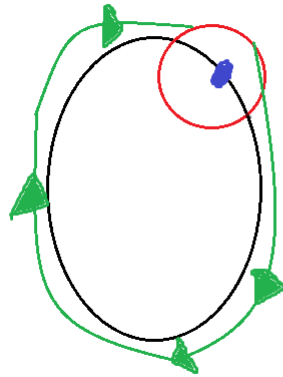
And the last step, namely filling in the area bounded by the resulting shape after pushing the circle around (**note:** we're not filling in the *inside*; rather, we're just filling in the "boundary" spanned by the boundary of the red circle):

Exercise 3.3. Let T^2 denote the torus $S^1 \times S^1$. What is $\pi_1 T^2$? More generally, if T^n denotes the n-torus $S^1 \times \cdots \times S^1$, what do you think $\pi_1 T^n$ is?

Exercise 3.4. Convince yourself that $\pi_1(X, x)$ is the collection of basepoint preserving maps from the circle $S^1 \to X$.

Recall that we had said earlier that "Two homotopy equivalent spaces have the 'same number of holes". How do we make this precise?

Let $S^1 \vee S^1$ denote the wedge product of two circles, which, in other words, means we join the two circles at a single point. Then $\pi_1(S^1 \vee S^1)$ is the *free group on* 2 *generators* a, b. This is the collection of all "words" one can write using a, b, a^{-1}, b^{-1}, such that we can simply "cancel" aa^{-1}, $a^{-1}a$, bb^{-1}, and $b^{-1}b$.

Exercise 3.5. Show that this is true using Exercise 3.2. Show also that $S^1 \vee S^1$ is homotopy equivalent to $\mathbf{R}^2$ with two points removed. More generally, convince yourself that π_1 of $\mathbf{R}^2$ with n points removed is isomorphic to the free group on n generators.

Exercise 3.6. Let S^2 denote the 2-sphere; this is simply a the boundary of a ball. What is $\pi_1 S^2$? Think intuitively.