# Special Relativity - Session 2
# Math Circle

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**Problem 1**  The *spacetime interval* $\Delta s$ for a given observer with coordinates $(x, t)$ is defined as

$$\Delta s^2 = -c^2(\Delta t)^2 + (\Delta x)^2$$

where $\Delta t$ and $\Delta x$ are the difference in the coordinates of two events. Check that general Lorentz transformations leave the spacetime interval invariant. In other words, check that

$$\Delta s'^2 = -c^2(\Delta t')^2 + (\Delta x')^2 = -c^2(\Delta t)^2 + (\Delta x)^2 = \Delta s^2$$

where

$$\Delta x' = \gamma(\Delta x - v\Delta t) \qquad\qquad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$c\Delta t' = \gamma\big(c\Delta t - \tfrac{v}{c}\Delta x\big).$$

As we explained, the space-time coordinates $(x, t)$ and $(x', t')$ are just labels for events happening in spacetime. Being different for different observers they do not have any physical meaning. However, the spacetime interval is invariant, that is, the same for any observer. And hence it can be used to define a sort of "distance" between events that we call proper time $\Delta\tau$ as

$$\Delta\tau = \frac{1}{c}\sqrt{-\Delta s^2},$$

Note that this coincides with the coordinate time $\Delta t$ in the rest frame since $\Delta x = 0$,

$$\Delta\tau = \frac{1}{c}\sqrt{-\Delta s^2} = \frac{1}{c}\sqrt{c^2(\Delta t)^2 - (\Delta x)^2} = \frac{1}{c}\sqrt{c^2(\Delta t)^2} = \frac{1}{c}c\Delta t = \Delta t$$

and it is in that sense that we call it a time. But of course in a different frame $\Delta x \neq 0$ and hence it would not be a time. Nevertheless, this is the only "physical time" that is relevant for an object or an observer, because it can only be *measured* by that object/observer. However, because the interval is invariant, it can be *calculated* by anybody else using the interval.

**Lemma**   Time Dilation Effect. Suppose Alice travels relative to Bob at a speed $v$. Suppose time $\tau$ is measured by Alice and time $t$ is measured by Bob. Then

$$t = \gamma\tau, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}$$

**Problem 2**   A muon is moving with speed $v = 0.999c$ vertically down through the atmosphere. If its half-life in its own reference frame is $1.5\mu s = 1.5 \cdot 10^{-6}s$ , what is the half-life as measured by an observer on Earth? Hint: use the time dilation lemma.

Notice that if its half-life were to be the same as measured by an observer on Earth, then the muon could only travel $1.5\mu s \cdot 0.999c \approx 450$m before it decays, but muons are usually produced when cosmic rays bombard the upper atmosphere about 10 km above the ground. This would mean we could never detect muons on the ground. Fortunately, we have Special Relativity.

**Problem 3**   Two twins are born on planet earth. Immediately after the birth, one of them is taken on a space trip to a star 10 light years away from earth, and back, in a spaceship that can travel with velocity $v = \frac{3}{5}c$. Draw the spacetime diagram in the earth/staying twin frame, in particular draw the trajectories of the earth, the star, and the spaceship.

Now draw the spacetime diagram from the point of view of the travelling twin, in which the earth flies away and back in the opposite direction with velocity $v = -\frac{3}{5}c$, and the star flies towards the observer and then away with the same velocity.

Using the spacetime interval calculate the proper time from when they part until they meet. That is, working in the frame of the twin at rest, calculate $\Delta s^2$ and convert to $\Delta\tau$ for each leg of the travelling twin's trip. Then add $\Delta\tau$ from the two legs, which should give the total proper time for the travelling twin's journey. Then calculate $\Delta\tau$ for the stationary twin. You'll find that the two twins have different proper times. (We'll explain this below.) How would you expect these calculations to come out if you worked in the frame of the travelling twin?

Imagine each twin has a clock which they always carry with them. Since this clock is always in the rest frame, it will measure the proper time for that twin. According to the principle of relativity, physics should be the same in different reference frames. This suggests that the two clocks should agree when the travelling twin returns to Earth, but as you calculated above, the proper times are different, so the clocks will *not* agree. This appears to be a violation of the principle of relativity!

Before you worry about this too much, recall that the principle of relativity actually states that the laws of physics should be the same for *inertial* observers. This means that in order for the principle to hold, the observers cannot be accelerating with respect to one another. But the travelling twin necessarily accelerates during the trip to the star, especially when the rocket turns around and comes back to earth. Thus, the principle of relativity no longer holds, and we have no reason to be alarmed that the twins disagree about how much time has passed in their rest frames.