Trigonometry

ORMC

05/19/24

1 Putnam and Beyond Problems

Problem 1.1. Find the range of the function $f(x) = (\sin(x) + 1)(\cos(x) + 1)$.

Hint: Turn this into a function of $\sin(x) + \cos(x)$, and find the range of that.

Problem 1.2. Prove that for all integers $n \geq 0$ and all $x \in \left(0, \frac{\pi}{2}\right)$,

$$\sec^{2n}(x) + \csc^{2n}(x) \geq 2^{n+1}.$$

Hint: Combine basic trig identities with AM-GM.

1.1 Substitution

Often, algebra problems (particularly those involving square roots) are easier to solve if we think of our variables as trigonometric functions of other variables. This can give a geometric flavor to the problem, and allows us to use the algebra of trig identities.

Problem 1.3. Prove that

$$\sqrt{ab} + \sqrt{(1-a)(1-b)} \leq 1.$$

Hint: Let a and b be functions of some angles α and β respectively, so that the left side turns into a trig function of α and β that is definitely at most 1.

Problem 1.4. Let a, b, c be real numbers. Prove that

$$(ab + bc + ca - 1)^2 \leq (a^2 + 1)(b^2 + 1)(c^2 + 1).$$

Hint: The expression $x^2 + 1$ should basically always make you think "tangent".

Problem 1.5. Prove that

$$27 \sin^3 9^\circ + 9 \sin^3 27^\circ + 3 \sin^3 81^\circ + \sin^3 243^\circ = 20 \sin 9^\circ$$

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Hint: Find a triple-angle formula for $\sin(3x)$.

2 Competition Problems

Problem 2.1 (BAMO 2014 Problem 3)**.** Suppose that for two real numbers x and y the following equality is true:

$$(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = 1.$$

Find (with proof) the value of $x + y$.

Problem 2.2 (BAMO 2009 Problem 7)**.** Let $\triangle ABC$ be an acute triangle with angles α, β, γ. Prove that

$$\frac{\cos\alpha}{\cos(\beta-\gamma)} + \frac{\cos\beta}{\cos(\gamma-\alpha)} + \frac{\cos\gamma}{\cos(\alpha-\beta)} \geq \frac{3}{2}$$

Problem 2.3. The area of square $ABCD$ is 196. Point E is inside the square, at the same distances from points D and C, and such that $\angle DEC = 150°$. What is the perimeter of $\triangle ABE$ equal to?

Problem 2.4 (USAMO 1992 Problem 2)**.** Prove

$$\frac{1}{\cos 0° \cos 1°} + \frac{1}{\cos 1° \cos 2°} + \cdots + \frac{1}{\cos 88° \cos 89°} = \frac{\cos 1°}{\sin^2 1°}.$$

Geometric hint: Construct a triangle, broken up into 89 pieces, such that the area of each piece is proportional to a term of the sum.

Algebraic hint: Make this a telescoping sum.

Problem 2.5 (USAMO 1998 Problem 3)**.** Let $a_0, \cdots a_n$ be real numbers in the interval $\left(0, \frac{\pi}{2}\right)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1$$

Prove that $\tan(a_0)\tan(a_1)\cdots\tan(a_n) \geq n^{n+1}$.

Hint: You'll want to use one of the most classic inequalities, and just a little bit of tangent addition formula. To simplify your calculations, let $y_i = \tan\left(a_i - \frac{\pi}{4}\right)$.

Problem 2.6 (USAMO 1980 Problem 3)**.** $A + B + C$ is an integral multiple of π. x, y, and z are real numbers. If $x\sin(A) + y\sin(B) + z\sin(C) = x^2\sin(2A) + y^2\sin(2B) + z^2\sin(2C) = 0$, show that $x^n\sin(nA) + y^n\sin(nB) + z^n\sin(nC) = 0$ for any positive integer n.

Hint: Make this a question about three complex numbers, and use the fact that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$